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A CONSTRUCTION OF ALL NORMAL SUBGROUP LATTICES OF 2-TRANSITIVE AUTOMORPHISM GROUPS OF LINEARLY ORDERED SETS

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ABSTRACT

We give a complete classification and construction of all normal subgroup lattices of 2-transitive automorphism groups $A(\Omega)$ of linearly ordered sets $(\Omega, \leqq)$. We also show that in each of these normal subgroup lattices, the partially ordered subset of all those elements which are finitely generated as normal subgroups forms a lattice which is closed under even countably-infinite intersections, and we derive several further group-theoretical consequences from our classification.

§1. Introduction

An infinite linearly ordered set ("chain") $(\Omega, \leqq)$ is called doubly homogeneous, if its automorphism group, i.e. the group of all order-preserving permutations, $A(\Omega) \equiv \mathrm{Aut}((\Omega, \leqq))$, acts 2-transitively on it. Chains $(\Omega, \leqq)$ of this type and certain normal subgroups of their automorphism groups $A(\Omega)$ have been used, e.g., for the construction of infinite simple torsion-free groups (Higman [8]) or, in the theory of lattice-ordered groups (l-groups), in dealing with embeddings of arbitrary l-groups into simple divisible l-groups (Holland [9]); for a variety of further results see Glass [7]. In this paper we classify and construct all normal subgroup lattices $N(A(\Omega))$ of the groups $A(\Omega)$, if $(\Omega, \leqq)$ is a doubly homogeneous chain. Our classification does not assume the class of all doubly homogene-

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ous chains $(\Omega, \leqq)$ to be given. As a main consequence we obtain that in each of these lattices $N(A(\Omega))$, the partially ordered set of all those elements which are finitely generated as normal subgroups forms a lattice (under inclusion) which is even closed under countable intersections. Furthermore, we obtain several structure results on $N(A(\Omega))$ first proved in [5], [2] again as consequences of our analysis (see Corollaries 2.14, 2.15), and we solve an open problem of [5; p. 124] (cf. Corollary 3.16).

Let $(\Omega, \leqq)$ be a doubly homogeneous chain in the following. Obvious normal subgroups of $A(\Omega)$ are $R(\Omega)$ $(L(\Omega))$, the group of all automorphisms with support bounded on the left (right), respectively, and $B(\Omega) = R(\Omega) \cap L(\Omega)$. According to Higman [8], $B(\Omega)$ is always simple and contained in every non-trivial normal subgroup of $A(\Omega)$. Holland [9] and Lloyd [11] showed that if the cofinality $\operatorname{cof}(\Omega)$ (coinitiality $\operatorname{coi}(\Omega)$) of Ω is countable, then the group $R(\Omega)/B(\Omega)$ $(L(\Omega)/B(\Omega))$ is simple. Contradicting [12], Ball [1] presented a certain class of doubly homogeneous chains Ω with $\operatorname{cof}(\Omega) = \aleph_1$ where $R(\Omega)/B(\Omega)$ is not simple. In [5], one of the present authors showed for any doubly homogeneous chain Ω that if $\operatorname{cof}(\Omega) \neq \aleph_0$, then $R(\Omega)/B(\Omega)$ has indeed uncountably many normal subgroups; furthermore, any subnormal subgroup of $A(\Omega)$ is normal. Moreover, in [5] the lattice $N(A(\Omega))$ was shown to be isomorphic to a certain lattice depending only on the structure of the Dedekind-completion $(\bar{\Omega}, \leqq)$ of $(\Omega, \leqq)$ (cf. also Ball and Droste [2]). Further group-theoretical properties of the lattice $N(A(\Omega))$ will be contained in [6].

Our main result mentioned above and consequences are stated explicitly in §2. Here we construct to each doubly homogeneous chain Ω a pair of trees of a certain type defined independently of $(\Omega, \leqq)$ and to these trees a corresponding lattice, which is shown to be isomorphic to the partially ordered subset of $N(A(\Omega))$ consisting of all finitely generated non-trivial normal subgroups of $A(\Omega)$. The proof, which is contained in §3, uses the characterization of [5] of the lattice $N(A(\Omega))$ mentioned above and is based on an analysis of certain convergence properties of fixed point sets of automorphisms of Ω. In order to be self-consistent, all the background results developed in [5] (see also [2]) which are employed here are reviewed. Then in §4 we prove conversely that each tree of the given type can be realized through a doubly homogeneous chain $(\Omega, \leqq)$.

§2. The main results

Recall that a tree $(T, \leqq)$ is a partially ordered set with a smallest element such that for each $a \in T$ the set $\{b \in T \mid b \leqq a\}$ is well-ordered. Linearly ordered sets will also be called chains. If $(T, \leqq)$ is a tree and $\varnothing \neq A \subseteq (T, \leqq)$ a chain, an element $x \in T$ is a strict upper bound of A if $x > a$ for all $a \in A$; let M_A always be the set of all minimal strict upper bounds of A. We also write $M_a = M_{\{a\}}$ for $a \in T$. Hence $M_a = \varnothing$ iff a is a maximal element in T. As usual, in the following cardinals are identified with the least ordinal of their cardinality. For a mapping f, let a^f denote its value at a.

DEFINITION 2.1. We will consider trees $(T, \leqq)$ with a mapping $\Phi: T \to \{1,2,3\}$ and order-relations $\leqq_a$ on M_a for each $a \in T$ with $M_a \neq \varnothing$ such that the following conditions are satisfied:

(1) If $a \in T$ and $a^\Phi = 2$, then $M_a = \varnothing$ and the set $\{x \in T \mid x < a\}$ is non-empty and contains no maximal element.

(2) If $a \in T$ and $M_a \neq \varnothing$, then

(i) $|M_a|$ is a regular uncountable cardinal;

(ii) $(M_a, \leqq_a)$ is well-ordered (inversely well-ordered) and isomorphic (anti-isomorphic) to $|M_a|$, if $a^\Phi = 3$ ($a^\Phi = 1$), respectively;

(iii) $x^\Phi = 1$ ($x^\Phi = 3$) for all $x \in M_a$ if $a^\Phi = 3$ ($a^\Phi = 1$), respectively.

(3) If $\varnothing \neq P \subseteq (T, \leqq)$ is a chain which contains no maximal element, then either (i) $|M_P| = 1$ and $M_P^\Phi = \{2\}$,
or (ii) $|M_P| = 3$ and $M_P^\Phi = \{1,2,3\}$.

NOTATION. Let $\mathrm{Mid}(T) = \{a \in T \mid a^\Phi = 2,\ |M_{\{x \in T \mid x < a\}}| = 3\}$, the set of all "middle" points of T (in a sense explained later), and $T^\sim = T \setminus \mathrm{Mid}(T)$.

Let $\mathcal{T}_R$ ($\mathcal{T}_L$) be the class of all trees $(T, \leqq)$ satisfying these conditions and, in addition, $(\min T)^\Phi = 3$ ($(\min T)^\Phi = 1$). We put $\mathcal{T} = \mathcal{T}_R \cup \mathcal{T}_L$.

Note that any $T \in \mathcal{T}$ has either precisely one or uncountably many elements.

DEFINITION 2.2 of a filter $B(T)$ on T for $T \in \mathcal{T}$
For each $a \in T$ with $M_a \neq \varnothing$, let

$$B_a = \{C \subseteq M_a \mid C \text{ is closed and unbounded above (below) in } M_a\},$$

if $(M_a, \leqq_a)$ is well-ordered (inversely well-ordered), respectively. We call a subset $B \subseteq T$ big, if there is a set $A \subseteq B$ which satisfies $\min(T) \in A$ and the following three conditions:

(i) if $a \in A$, then $\{x \in T \mid x < a\} \subseteq A$;

(ii) if $a \in A$ with $M_a \neq \varnothing$, then $A \cap \dot{M}_a \in B_a$;

(iii) if $a \in T$, $X = \{x \in T \mid x < a\} \subseteq A$, and X is non-empty and contains no maximal element, then $M_X \subseteq A$.

Let $B(T)$ be the set of all big subsets of T.

It is easy to see that $B(T)$ is a filter on the set T (i.e., in the power set of T) and, in particular, closed under countable intersections, since each B_a ($a \in T, M_a \neq \emptyset$) has this property.

NOTATION. Let $T \in \mathcal{T}$, $a \in \mathrm{Mid}(T)$ and $M_{\{x\in T \mid x<a\}} = \{y, a, z\}$ with $y^\Phi = 1$, $a^\Phi = 2$, $z^\Phi = 3$. Then we also write $y = a_1$, $z = a_3$.

DEFINITION 2.3. For $T \in \mathcal{T}$, let T^+ be the set of all subsets A of T satisfying the following conditions:

(i) $\{x \in T \mid x < a\} \subseteq A$ for each $a \in A$;

(ii) if $a \in \mathrm{Mid}(T)$ and $A \cap \{a_1, a_3\} \neq \emptyset$, then $a \in A$;

(iii) if $a \in T^-$, $a^\Phi = 2$ and $\{x \in T \mid x < a\} \subseteq A$, then $a \in A$.

Note that $(T^+, \subseteq)$ is a partially ordered set closed under arbitrary intersections, hence $B(T) \cap T^+$ is a filter in $(T^+, \subseteq)$ which is closed under countable intersections. Furthermore, for each $A \subseteq T$ satisfying the conditions (i)–(iii) of (2.2) we have $A \in T^+$.

DEFINITION 2.4 of the partially ordered set $(T^*, \leqq)$ for $T \in \mathcal{T}$

For two subsets $X, Y \in T^+$ we put $X \leqq Y$ if there exists a $B \in B(T) \cap T^+$ — or, equivalently, $B \in B(T)$ — with $X \cap B \subseteq Y$, and $X \sim Y$ if $X \leqq Y$ and $Y \leqq X$. Let $[X] = \{Y \in T^+ \mid X \sim Y\}$ be the equivalence class of $X \in T^+$ with respect to $\sim$, and let $T^* = \{[X] \mid X \in T^+\}$. We define $[X] \leqq [Y]$ for $X, Y \in T^+$ by $X \leqq Y$. Then $(T^*, \leqq)$ is a partially ordered set which is closed under countable infima by the remarks following Definition 2.3.

Clearly, $[\emptyset]$ is the smallest and $[T]$ the largest element of T^*, and $[\{\min T\}]$ is the smallest element of $T^* \setminus \{[\emptyset]\}$. Moreover, if T has only one element, then $B(T) = \{T\}$, $T^+ = \{\emptyset, T\}$ and $|T^*| = 2$. Conversely, $|T^*| = 2$ implies $|T| = 1$. Next we give a further still easy but non-trivial and important

EXAMPLE 2.5. Assume that each maximal chain in $T \in \mathcal{T}$ has precisely two elements. Then $T = T^- = \{a\} \dot\cup M_a$ where $a = \min T$ and $(M_a, \leqq_a)$ is well-ordered (inversely well-ordered) and isomorphic (anti-isomorphic) to $\kappa = |M_a| = |T|$, a regular uncountable cardinal, if $T \in \mathcal{T}_R$ ($T \in \mathcal{T}_L$), respectively. A subset $B \subseteq T$ is big iff $a \in B$ and $B \cap M_a \supseteq C$ for some $C \in B_a$. Also, $T^+ = \{A \subseteq T \mid a \in A\} \cup \{\emptyset\}$, and for $A, B \in T^+ \setminus \{\emptyset\}$ we have $[A] \leqq [B]$ in T^* iff $A \cap C \subseteq B$ for some $C \in B_a$. Thus $(T^* \setminus \{[\emptyset]\}, \leqq)$ is isomorphic to the Boolean algebra $A(\kappa) = \mathcal{P}(\kappa)/\mathcal{I}(\kappa)$, where here $\mathcal{P}(\kappa)$ is the power set of κ and $\mathcal{I}(\kappa)$ the ideal of "thin" subsets of κ (a subset $A \subseteq \kappa$ is called "thin", if $A \cap C = \emptyset$ for some subset $C \subseteq \kappa$ which is closed and unbounded above in κ;

cf. [10; p. 58]). In particular, under the assumption of this example the structure of $(T^*, \leqq)$ is uniquely determined by $\kappa = |T|$.

If Ω is an infinite chain and $k \in \mathbf{N}$, $A(\Omega)$ is called k-transitive if for any subsets $A, B \subseteq \Omega$ with $|A| = |B| = k$ there exists $\alpha \in A(\Omega)$ such that $A^\alpha = B$. The following remark is well-known.

REMARK 2.6. Let $(\Omega, \leqq)$ be an infinite chain. The following are equivalent:
(1) $A(\Omega)$ is k-transitive for some $k \in \mathbf{N}$ with $k \geqq 2$.
(2) $A(\Omega)$ is k-transitive for any $k \in \mathbf{N}$ with $k \geqq 2$.
(3) Ω is unbounded, and any two intervals $[a, b]_\Omega, [c, d]_\Omega$ $(a, b, c, d \in \Omega$, $a < b$, $c < d)$ are order-isomorphic.
If one of these conditions is satisfied, $(\Omega, \leqq)$ is called *doubly homogeneous.*

For a group G, let $(\alpha) = \bigcap \{N \mid \alpha \in N \lhd G\}$, the normal subgroup of G generated by $\alpha \in G$, $N(G) = \{N \mid \{1\} \neq N \lhd G\}$ the set of all *non-trivial* normal subgroups of G, and

$$N_1(G) = \{(\alpha) \mid \alpha \in G \backslash \{1\}\} \subseteq N(G),$$

the set of all non-trivial normal subgroups of G which are generated by a single element. Then we have the following

PROPOSITION 2.7 ([5: Prop. 6.3]). *Let Ω be a doubly homogeneous chain. Then $(N_1(A(\Omega)), \subseteq)$ is a join-semilattice, i.e. $(\alpha)\cdot(\beta) \in N_1(A(\Omega))$ for all $\alpha, \beta \in A(\Omega)\backslash\{1\}$, and $(N(A(\Omega)), \subseteq)$ is a complete algebraic (in particular distributive) lattice isomorphic to the lattice of all ideals of $(N_1(A(\Omega)), \subseteq)$.*

Because of this result, which follows immediately from [3; Theorem XIII 18] and [7; Theorem 2.3.1], $N_1(A(\Omega))$ coincides with the set of all compact elements of the complete algebraic lattice $(N(A(\Omega)), \subseteq)$ (cf. [3; Theorem VIII 8]), i.e. with the set of all elements of $N(A(\Omega))$ which are finitely generated as normal subgroups, and it suffices to examine the structure of the semilattice $N_1(A(\Omega))$. If $(A, \leqq)$, $(B, \leqq)$ are partially ordered sets, let $(A, \leqq) \times (B, \leqq)$ be the set $A \times B$ together with a partial ordering defined by $(a, b) \leqq (a', b')$ iff $a \leqq a'$ and $b \leqq b'$ $(a, a' \in A,\ b, b' \in B)$. We now come to our

MAIN THEOREM 2.8. *Up to isomorphism, the partially ordered sets $(N_1(A(\Omega)), \subseteq)$, where Ω ranges through all doubly homogeneous chains, are precisely the partially ordered sets $(T_l^*, \leqq) \times (T_r^*, \leqq)$, where $T_l \in \mathcal{T}_L$, $T_r \in \mathcal{T}_R$. Moreover, these partially ordered sets are lattices which are closed under countable infima.*

This theorem follows immediately from Theorems 2.9–2.11 given below which even sharpen (2.8). We proceed as follows. In Construction 2.16 we will define for each doubly homogeneous chain Ω two trees $T_r(\Omega) \in \mathcal{T}_R$, $T_l(\Omega) \in \mathcal{T}_L$. Then we show:

THEOREM 2.9. *Let Ω be a doubly homogeneous chain and $T_l = T_l(\Omega)$, $T_r = T_r(\Omega)$. Then the partially ordered sets $(N_1(R(\Omega)), \subseteq)$ and $(T_r^*, \leqq)$, $(N_1(L(\Omega)), \subseteq)$ and $(T_l^*, \leqq)$, and $(N_1(A(\Omega)), \subseteq)$ and $(T_l^*, \leqq) \times (T_r^*, \leqq)$, respectively, are isomorphic.*

Here the fact that $T_r^*, T_l^*, T_l^* \times T_r^*$ have smallest elements is reflected by Higman's result (see (2.13)) that $B(\Omega)$ is the smallest element of $N_1(R(\Omega))$, $N_1(L(\Omega))$, $N_1(A(\Omega))$, respectively. The proof of Theorem 2.9 is given in §3. It heavily uses methods developed in [5] (cf. also [2]) concerning fixed point sets of automorphisms $\alpha \in A(\Omega)$. The necessary background results are reviewed at the beginning of §3.

COROLLARY 2.10. *Let Ω be a doubly homogeneous chain. Then $(N_1(A(\Omega)), \subseteq)$, $(N_1(R(\Omega)), \subseteq)$, and $(N_1(L(\Omega)), \subseteq)$ are lattices which are closed under countable infima. $N_1(A(\Omega))$ is closed under countable intersections.*

PROOF. It suffices to show the first assertion, since this implies the final statement of the corollary. By Proposition 2.7, the three partially ordered sets under examination are join-semilattices. Hence the result follows from $T_l(\Omega) \in \mathcal{T}_L$, $T_r(\Omega) \in \mathcal{T}_R$, Theorem 2.9, and Definition 2.4.

Note that the simple group-theoretical result that $N_1(A(\Omega))$ is closed under (even countably-infinite) intersections appears here as a consequence of our set-theoretic characterization of $(N_1(A(\Omega)), \subseteq)$. In fact, we have no other proof for this result.

Conversely, in §4 we show that each tree $T \in \mathcal{T}$ can be realized through a doubly homogeneous chain Ω:

THEOREM 2.11. *Let $T_L \in \mathcal{T}_L$, $T_R \in \mathcal{T}_R$. Then for any regular cardinal λ with $\lambda \geqq |T_L| + |T_R|$ there exists a doubly homogeneous chain $(\Omega, \leqq)$ of cardinality λ and trees $T_l(\Omega)$, $T_r(\Omega)$ such that $(T_L, \leqq_L) = (T_l, \leqq_l)$, $(T_R, \leqq_R) = (T_r, \leqq_r)$ and hence, in particular, $(N_1(A(\Omega)), \subseteq) \cong (T_L^*, \leqq) \times (T_R^*, \leqq)$. Moreover, the set $(\Omega, \leqq)$ can be chosen such that each point $a \in \Omega$ has countable coterminality.*

Here, an element $x \in \Omega$ has countable coterminality if there are countable subsets $A, B \subseteq \Omega$ such that $x = \sup A = \inf B$ and $a < x < b$ for all $a \in A$, $b \in B$. Before proceeding, let us fix some

NOTATION. $A \dot{\cup} B$, $\dot{\bigcup} A_i$ denote *disjoint* unions. If $(M, \leqq)$ is a partially ordered set, $A, B \subseteq M$ and $x, y \in M$, let $[x, y]_A = \{z \in A \quad x \leqq z \leqq y\}$, and write $A < B$ $(A \leqq B)$ iff $a < b$ $(a \leqq b)$ for all $a \in A$, $b \in B$, and $x \leqq A$ iff $\{x\} \leqq A$; A is called unbounded above (below) in M if there exists no $x \in M$ such that $A \leqq x$ $(x \leqq A)$, unbounded in M if A is unbounded above and below in M, and unbounded (unbounded above, below) if A is unbounded (unbounded above, below) in A. A is called dense if for all $a, b \in A$ with $a < b$ there exists $a \in A$, such that $a < c < b$.

Now let $(\Omega, \leqq)$ be a dense unbounded chain. We let $(\bar{\Omega}, \leqq)$ denote the Dedekind-completion of $(\Omega, \leqq)$ and $\hat{\Omega} = \bar{\Omega} \cup \{-\infty, \infty\}$ with $-\infty < x < \infty$ for all $x \in \bar{\Omega}$. We always consider Ω as a subset of $\hat{\Omega}$, and if we have to distinguish between different orders, $a \leqq b$ for elements $a, b \in \hat{\Omega}$ will always mean $a \leqq_{\hat{\Omega}} b$, i.e. with respect to the natural order of $\hat{\Omega}$. Whenever $a, b \in \hat{\Omega}$ with $a < b$, let $[a, b] = \{z \in \hat{\Omega} \mid a \leqq z \leqq b\}$. Similarly, if $A \subseteq \hat{\Omega}$, $\sup A$ and $\inf A$ are always taken in $(\hat{\Omega}, \leqq)$ if not specified explicitly otherwise.

We put $\operatorname{cof}_\Omega(a) = \operatorname{cof}_{\bar{\Omega}}(a) = \min\{|A| \mid A \subseteq \bar{\Omega},\ A < a,\ a = \sup A\}$ for $a \in \bar{\Omega} \cup \{\infty\}$, the cofinality of a, and we write $\operatorname{cof}(a)$ if there is no ambiguity about the chain Ω. Similarly we put $\operatorname{coi}(a) = \min\{|A| \mid a \subseteq \bar{\Omega},\ a < A,\ a = \inf A\}$ for each $a \in \bar{\Omega} \cup \{-\infty\}$, the coinitiality of a. If $a \in \bar{\Omega}$ and $\operatorname{cof}(a) = \operatorname{coi}(a)$, let $\operatorname{cot}(a) = \operatorname{cof}(a)$, the coterminality of a. Let $\operatorname{cof}(\Omega) = \operatorname{cof}(\infty)$, $\operatorname{coi}(\Omega) = \operatorname{coi}(-\infty)$, and $\operatorname{cot}(\Omega) = \operatorname{cof}(\Omega)$ if $\operatorname{cof}(\Omega) = \operatorname{coi}(\Omega)$. If $A \subseteq B \subseteq \hat{\Omega}$, A is called closed in B if $B \cap \{\sup C, \inf C\} \subseteq A$ for any $\varnothing \neq C \subseteq A$.

As is well-known, any doubly homogeneous chain is dense and unbounded.

Next we wish to give two examples for the construction of the tree $T_r(\Omega) \in \mathcal{T}_R$ which serve to obtain consequences of Theorem 2.9.

EXAMPLE 2.12. Let $(\Omega, \leqq)$ be a doubly homogeneous chain.

(a) For $T_r = T_r(\Omega)$, we will have $|T_r^*| = 2$ iff $T_r = \{\infty\}$ iff $\operatorname{cof}(\Omega) = \aleph_0$. In this case, $M_\infty = \varnothing$ and $\infty^\Phi = 3$. Similarly, $T_l(\Omega) = \{-\infty\}$ iff $\operatorname{coi}(\Omega) = \aleph_0$.

(b) Suppose $\kappa = \operatorname{cof}(\Omega) \neq \aleph_0$ and there exists a set $A \subseteq \bar{\Omega}$ such that A is closed and unbounded above in $\bar{\Omega}$ with $\operatorname{coi}(a) = \aleph_0$ for each $a \in A$. Choose a well-ordered subset $M \subseteq A$ such that $M \cong \kappa$ and M is closed and unbounded above in $\bar{\Omega}$. We put $T_r = T_r(\Omega) = M \dot{\cup} \{\infty\}$ and define a partial order $\leqq_r$ on T_r such that $a <_r b$ iff $a = \infty$, $b \in M$, for any $a, b \in T_r$. Hence in $(T_r, \leqq_r)$ we have $M_\infty = M$ and $M_a = \varnothing$ for each $a \in M$. Furthermore, put $\infty^\Phi = 3$, $a^\Phi = 1$ for each $a \in M$, and $(M_\infty, \leqq_\infty) = (M, \leqq_{\bar{\Omega}})$. In particular, $T_r \in \mathcal{T}_R$ and in T_r each maximal linearly ordered subset has precisely two elements. Hence by Example 2.5 we have $(T_r^* \backslash \{[\varnothing]\}, \leqq) \cong \mathcal{A}(\kappa)$.

Now we wish to use (2.12) to derive consequences of Theorem 2.9. First let us note some basic and well-known properties of $N(A(\Omega))$:

PROPOSITION 2.13. *Let Ω be a doubly homogeneous chain. Then:*

(a) *Any normal subgroup of $R(\Omega)$ or $L(\Omega)$ is also normal in $A(\Omega)$.*

(b) *$B(\Omega)$ is simple and the smallest element of $N(A(\Omega))$ and hence also of $N_1(A(\Omega))$, $N_1(R(\Omega))$, and $N_1(L(\Omega))$.*

Here (a) has been generalized in [5; 6.15]; indeed any subnormal subgroup of $A(\Omega)$ is normal in $A(\Omega)$.

As a first consequence we obtain the following important result already mentioned in the introduction:

COROLLARY 2.14 ([5; Satz 6.34, Kor. 6.25]). *Let Ω be a doubly homogeneous chain. Then:*

(a) $A(\Omega)$, $R(\Omega)$, $L(\Omega) \in N_1(A(\Omega))$.

(b) *There exist smallest normal subgroups $N_1, N_2, N_3 \lhd A(\Omega)$ satisfying $B(\Omega) \subsetneqq N_1 \subseteq R(\Omega)$, $B(\Omega) \subsetneqq N_2 \subseteq L(\Omega)$, and $N_3 \not\subseteq L(\Omega) \cup R(\Omega)$. We have $N_1, N_2, N_3 \in N_1(A(\Omega))$.*

(c) *$R(\Omega)/B(\Omega)$ is simple iff* $\operatorname{cof}(\Omega) = \aleph_0$.

(d) *$L(\Omega)/B(\Omega)$ is simple iff* $\operatorname{coi}(\Omega) = \aleph_0$.

(e) *$B(\Omega)$, $R(\Omega)$, and $L(\Omega)$ are all non-trivial proper normal subgroups of $A(\Omega)$ if and only if* $\operatorname{cot}(\Omega) = \aleph_0$.

Here in (c)–(e) the "if" part is due to Holland [9] and Lloyd [11]. In Ball [1], (a), (b), the non-simplicity of $R(\Omega)/B(\Omega)$ and the subsequent Corollary 2.15 have been proved under the special assumption that $\operatorname{cof}(\Omega) = \aleph_1$, $\operatorname{coi}(\Omega) = \aleph_0$, and there exists a subset $A \subseteq \bar{\Omega}$ which is closed and unbounded above in $\bar{\Omega}$ with $\operatorname{cot}(a) = \aleph_0$ for each $a \in A$.

PROOF. (a) This follows from Theorem 2.9 and the fact that $[T_r]$ ($[T_l]$, $([T_l],[T_r])$) is the largest element of $(T_r^*, \leqq)$ ($(T_l^*, \leqq)$, $(T_l^* \times T_r^*, \leqq)$), respectively.

(b), (c), (d). We prove the result for $R(\Omega)$. The partially ordered set $(T_r^*, \leqq)$ satisfies $[\varnothing] < [\{\infty\}] \leqq T_r^* \backslash \{[\varnothing]\}$. If $\psi : (T_r^*, \leqq) \to (N_1(R(\Omega)), \subseteq)$ is any isomorphism (which exists by (2.9)), then $[\varnothing]^\psi = B(\Omega)$ and $[T_r]^\psi = R(\Omega)$ by (a), and thus we may put $N_1 = [\{\infty\}]^\psi$. Then by (2.12)(a) we have $N_1 \subsetneqq R(\Omega)$ iff $[\{\infty\}] < [T_r]$ iff $\operatorname{cof}(\Omega) \neq \aleph_0$. Now a symmetry-argument implies the result for $L(\Omega)$. Finally, let $N_3 = N_1 \cdot N_2$.

(e) By (c) and (d).

As an immediate consequence of Theorem 2.9 and Example 2.12 we have

COROLLARY 2.15 ([5; Satz 7.14]; [1] if $\kappa = \aleph_1$). *Let Ω be a doubly homogeneous chain with $\kappa = \operatorname{cof}(\Omega) \neq \aleph_0$. Assume there exists a subset $A \subseteq \bar{\Omega}$ which is closed and unbounded above in $\bar{\Omega}$ such that $\operatorname{coi}(a) = \aleph_0$ for each $a \in A$. Then $(N_1(R(\Omega))\setminus\{B(\Omega)\}, \subseteq)$ is isomorphic to the Boolean algebra $\mathscr{A}(\kappa)$ defined in Example* 2.5.

For a generalization of this result see [5, 6].

NOTATION. If $(A_i, \leqq_i)$, $i = 1, 2$, are partially ordered sets, we write $(A_1, \leqq_1) \subseteq (A_2, \leqq_2)$ iff $A_1 \subseteq A_2$ and, for all $a, b \in A_1$, $a \leqq_1 b$ iff $a \leqq_2 b$. If an element a is the supremum of a set $Z \subseteq A_1$ with respect to $\leqq_1$ (and maybe not w.r.t. $\leqq_2$), we write $a = \sup_{A_1} Z$. If I is a set of ordinals, $L(I)$ denotes the set of all limit-ordinals belonging to I.

If $(T, \leqq)$ is a tree, we will call any chain $P \subseteq T$ which is unbounded above in T a *path in* T. Now we come to the already announced

CONSTRUCTION 2.16. For each doubly homogeneous chain Ω, we define a "right" tree $T_r = T_r(\Omega) \in \mathscr{T}_R$ and a "left" tree $T_l = T_l(\Omega) \in \mathscr{T}_L$.

CONSTRUCTION OF $(T_r, \leqq_r)$. By transfinite induction, we will define an index set I_r consisting of ordinals such that $i \in I_r$ whenever $0 \leqq i \leqq m = \max(I_r)$, and trees $(T_i, \leqq_i)$ with $T_i \subseteq \hat{\Omega}$ for each $i \in I_r$, $(T_i, \leqq_i) \subseteq (T_j, \leqq_j)$ if $i < j$ $(i, j \in I_r)$, and $(T_r, \leqq_r) := (T_m, \leqq_m)$. For each $i \in I_r$ and $a \in T_i$ we will choose an element $a' \in (\bar{\Omega}\setminus T_i) \cup \{a\}$; then we always put $V_a = [a', a]$ ($\{a\}, [a, a']$) if $a' < a$ ($a = a', a < a'$), respectively; furthermore, let $S_i = T_i \setminus \bigcup_{j<i} T_j$ and

$$Z_i = \{a \in S_i \mid a' < a, \operatorname{cof}(a) \neq \aleph_0, \text{or } a < a', \operatorname{coi}(a) \neq \aleph_0\}.$$

For each $i \in I_r$, $(T_i, \leqq_i)$ will satisfy the following property:

(*) For any $a, b \in T_i$ with $a \neq b$, we have either $V_a \cap V_b = \varnothing$, or $V_a \subseteq V_b \setminus \{b, b'\}$, or $V_b \subseteq V_a \setminus \{a, a'\}$; moreover, $V_b \subseteq V_a \setminus \{a, a'\}$ iff $a <_i b$. All elements of S_i are maximal in $(T_i, \leqq_i)$.

In particular, $b \in V_a \setminus \{a\}$ implies $a <_i b$ and $a \notin S_i$ $(a, b \in T_i)$.

First, let $T_0 = \{\infty\}$ and choose $\infty' \in \bar{\Omega}$ arbitrarily.

Now let $i \geqq 0$ and $(T_j, \leqq_j)$, $\{a' \mid a \in T_i\}$, S_i, Z_i already be defined $(j \leqq i)$ such that T_i satisfies (*). If $Z_i = \varnothing$, let $(T_r, \leqq_r) = (T_i, \leqq_i)$, $I_r = \{j \mid j \leqq i\}$, and our construction is finished. (For example, for $i = 0$ this is the case iff $\operatorname{cof}(\Omega) = \aleph_0$; cf. (2.12).) Now assume $Z_i \neq \varnothing$. For each $a \in Z_i$ with $a' < a$ note that $V_a \cap T_i = \{a\}$

by the conclusion to (*). Let $M_a \subseteq (a', a)$ be closed and unbounded above in (a', a) such that $(M_a, \leqq_{\bar\Omega}) \cong \mathrm{cof}(a)$, in particular, $(M_a, \leqq_{\bar\Omega})$ is well-ordered and $a = \sup_{\bar\Omega} M_a$. For each $x \in M_a$ choose an element $x' \in \bar\Omega$ with $x < x' < \min\{z \in M_a \mid x < z\}$. Similarly, if $a \in Z_i$ and $a < a'$, let $M_a \subseteq (a, a')$ be closed and unbounded below in (a, a') such that $(M_a, \leqq_{\bar\Omega})$ is anti-isomorphic to $\mathrm{coi}(a)$; in particular, $(M_a, \leqq_{\bar\Omega})$ is inversely well-ordered and $a = \inf_{\bar\Omega} M_a$. For each $x \in M_a$ choose $x' \in \bar\Omega$ with $\max\{z \in M_a \mid z < x\} < x' < x$. Put $S_{i+1} = \dot\bigcup_{a \in Z_i} M_a$, and let $(T_{i+1}, \leqq_{i+1})$ be the tree satisfying $T_{i+1} = T_i \dot\cup S_{i+1}$, $(T_i, \leqq_i) \subseteq (T_{i+1}, \leqq_{i+1})$, and $x <_{i+1} b$ for $x \in T_{i+1}$, $b \in M_a \subseteq S_{i+1}$ $(a \in Z_i)$ iff $x \in T_i$ and $x \leqq_i a$. Then for each $a \in Z_i$, the set M_a indeed coincides with the set of all minimal strict upper bounds of a in $(T_{i+1}, \leqq_{i+1})$.

Finally, let i be a limit ordinal and $(T_j, \leqq_j)$ and a', V_a $(a \in T_j)$ already be defined such that T_j satisfies (*) (with i replaced by j) for each $j < i$. First note that any path P of $\bigcup_{j<i} T_i$ is a maximal path if and only if P satisfies: Whenever $a \in P \cap T_j$ for $j < i$, $x \in T_j$ and $x <_j a$, then $x \in P$. Now for each maximal path P of $\bigcup_{j<i} T_i$ let $V_P = \bigcap_{a \in P} V_a \subseteq \bar\Omega$. Then $V_P \neq \emptyset$ by (*) and the Dedekind-completeness of $\bar\Omega$. Let $a_P, c_P \in \bar\Omega$ with $a_P \leqq c_P$ and $V_P = [a_P, c_P]$. If $a_P = c_P$, put $a'_P = a_P$ and $M^*_P = \{a_P\}$. If $a_P < c_P$, choose $a'_P, b_P, c'_P \in \bar\Omega$ such that $a_P < a'_P < b_P < c'_P < c_P$, and let $b'_P = b_P$, $M^*_P = \{a_P, b_P, c_P\}$. Note that these elements are well-defined since if Q is another maximal path in $\bigcup_{j<i} T_j$ $(Q \neq P)$, then $V_Q \cap V_P = \emptyset$ by (*). Put

$$S_i = \dot\bigcup \{M_P \mid P \text{ maximal path in } \bigcup_{j<i} T_j\},$$

and let $(T_i, \leqq_i)$ be the tree satisfying $T_i = \bigcup_{j<i} T_j \dot\cup S_i$ and $a <_i b$ for $a, b \in T_i$ iff $a, b \in T_j$ and $a <_j b$ for some $j < i$, or else $a \in \bigcup_{j<i} T_j$, $b \in M_P \subseteq S_i$ for some maximal path $P \subseteq \bigcup_{j<i} T_j$ with $a \in P$. Note that if there exist no paths $P \subseteq \bigcup_{j<i} T_j$, we get $S_i = \emptyset$ and $\bigcup_{j<i} T_j = T_i = T_r$.

After at most $|\bar\Omega|$ steps our construction of $T_r = (T_r, \leqq_r)$ will be finished.

Notation. If $i \in L(I_r)$ and Q is a path in $\bigcup_{j<i} T_j$, let P be the uniquely determined maximal path of $\bigcup_{j<i} T_j$ containing Q. We put $V_Q = V_P$, $a_Q = a_P$, $a'_Q = a'_P, \ldots, M^*_Q = M^*_P$. Hence now we have $M^*_Q = M_Q$ in $(T_r, \leqq_r)$ for any path Q in $\bigcup_{j<i} T_j$ $(i \in L(I_r))$.

Construction of $(T_l, \leqq_l)$. Exactly as for $(T_r, \leqq_r)$; we only start with $T_0 = \{-\infty\}$ and $(-\infty)' \in \bar\Omega$ such that $(-\infty)' < \infty'$.

Definition 2.17. For $T = T_r(\Omega)$ or $T = T_l(\Omega)$, we always define $\Phi: T \to \{1,2,3\}$ by $a^\Phi = 3$ $(2,1)$ $[a \in T]$ if $a' < a$ $(a' = a, a < a')$, respectively, and we put $(M_a, \leqq_a) = (M_a, \leqq_{\bar{\Omega}})$ if $a \in T$, $M_a \neq \emptyset$. Then $T_r(\Omega) \in \mathcal{T}_R$, $T_l(\Omega) \in \mathcal{T}_L$.

Note that if $T = T_r(\Omega)$ or $T = T_l(\Omega)$, $a \in \mathrm{Mid}(T)$, $X = \{x \in T \mid x <_T a\}$, and $M_X = \{a_X, b_X, c_X\}$ with $a_X^\Phi = 1$, $b_X^\Phi = 2$, $c_X^\Phi = 3$, then $b_X = a$ and $a_X = \min_{\bar{\Omega}}(M_X) = a_1$, $c_X = \max_{\bar{\Omega}}(M_X) = a_3$ according to our notation. In this case we have

$$a_1 = \sup_{\bar{\Omega}}\{x \in X \mid x^\Phi = 1\} \quad \text{and} \quad a_3 = \inf_{\bar{\Omega}}\{x \in X \mid x^\Phi = 3\}.$$

It seems justified to call $a \in \mathrm{Mid}(T)$ a middle point of T since it was chosen "in the middle" of the non-trivial interval $[a_1, a_3]$, i.e. $a_1 < a < a_3$. Also, note that if $a \in T^{\sim}$ with $a^\Phi = 2$, then

$$a = \sup_{\bar{\Omega}}\{x \in T \mid x <_T a, x^\Phi = 1\} = \inf_{\bar{\Omega}}\{x \in T \mid x <_T a, x^\Phi = 3\}.$$

Finally, for any $a \in T$ we have $M_a = \emptyset$ iff one of the following three (mutually exclusive) conditions holds:

(i) $a' < a$, $\mathrm{cof}(a) = \aleph_0$, or $a < a'$, $\mathrm{coi}(a) = \aleph_0$;

(ii) $\{a\} = M_P$ where $P = \{x \in T \mid x <_T a\}$ is unbounded above;

(iii) $a \in \mathrm{Mid}(T)$.

It is easy to see that the tree $T_r = T_r(\Omega)$ as constructed above is determined uniquely iff $\mathrm{cof}(\Omega) = \aleph_0$ and uniquely up to isomorphism iff $\mathrm{cof}(\Omega) = \aleph_0$ or $\mathrm{coi}(a) = \aleph_0$ for each $a \in \bar{\Omega}$. However, the structure of the set $(T_r^*, \leqq)$ is always independent of the choices which were possible in Construction 2.16. This follows immediately from Theorem 2.9, but for the sake of completeness it seems appropriate to include a sketch of a direct proof (Remark 2.18 will not be needed later). Analogous remarks hold, of course, also for $T_l(\Omega)$.

REMARK 2.18. Let $T_k = T_{r,k}(\Omega)$ $(k = 1, 2)$ be two "right" trees of Ω constructed according to the requirements in (2.16). Then $(T_1^*, \leqq) \cong (T_2^*, \leqq)$.

PROOF (*sketch*). For $k = 1, 2$, let I_k be the index set corresponding to T_k, and for any $\emptyset \neq A \subseteq T_k$, let M_A^k be the set of all minimal strict upper bounds of A in T_k. We define a set $W_k \subseteq T_k$ by transfinite induction. Put $W_{k,0} = \{\infty\}$. If $i \in I_k$ and $W_{k,i} \subseteq T_{k,i}$ is already defined, let

$$V_{k,i+1} = \dot{\bigcup} \{M_a^1 \cap M_a^2 \mid a \in W_{k,i} \cap Z_{1,i} \cap Z_{2,i}\} \quad \text{and} \quad W_{k,i+1} = W_{k,i} \dot{\cup} V_{k,i+1}.$$

If $i \in L(I_k)$ and $W_{k,j} \subseteq T_{k,j}$ are already defined for each $j < i$, let

$$V_{k,i} = \dot{\bigcup} \{M_P^k \mid P \subseteq \bigcup_{j<i} {}^{\cdot}T_{k,j} \text{ path with } P \subseteq \bigcup_{j<i} W_{k,j}\} \quad \text{and} \quad W_{k,i} = \bigcup_{j<i} W_{k,j} \dot{\cup} V_{k,i}.$$

We put $W_k = \bigcup_{j \in I_k} W_{k,j}$. Then $V_{1,i+1} = V_{2,i+1}$ for each $i \in I_k$, $W_1 \cap T_2^- = W_2 \cap T_1^-$, and W_k satisfies conditions (2.2)(i)–(iii) for T_k, hence $W_k \in B(T_k) \cap T_k^+$ $(k = 1,2)$. We define a bijection $f: W_1 \to W_2$ by putting $a^f = b$ whenever either $a = b \in W_1 \cap W_2$ or $a \in W_1 \cap \mathrm{Mid}(T_1)$, $b \in W_2 \cap \mathrm{Mid}(T_2)$ satisfy $\{x \in T_1 \mid x <_{T_1} a\} = \{x \in T_2 \mid x <_{T_2} b\}$. Then the map $F: T_1^* \to T_2^*$, defined by $[A]^F = [B]$ whenever $(A \cap W_1)^f = B \cap W_2$ for $A \in T_1^+$, $B \in T_2^+$, establishes the required isomorphism.

It remains to prove Theorem 2.9 and Theorem 2.11. This will be done in §3 and §4, respectively. We remark that we keep the notation developed in this section for the rest of the paper.

§3. Analysis of $N_1(A(\Omega))$

This section is mainly devoted to the proof of Theorem 2.9. However, we will also obtain some further structure results on $N_1(A(\Omega))$ for doubly homogeneous chains Ω. For the convenience of the reader, we first summarize some background results developed in [5; §6] (cf. also [2]) which we will need here.

Notation. Let Ω be a dense unbounded chain. For any set $\emptyset \neq A \subseteq \hat{\Omega}$ we put

$$\underset{\rightarrow}{\lim}(A) = \{a \in \bar{\Omega} \cup \{\infty\} \mid a = \sup\{x \in A \mid x < a\},$$
$$\underset{\leftarrow}{\lim}(A) = \{a \in \bar{\Omega} \cup \{-\infty\} \mid a = \inf\{x \in A \mid a < x\},$$
$$\lim(A) = \underset{\rightarrow}{\lim}(A) \cup \underset{\leftarrow}{\lim}(A),$$

and we call A closed upwards (downwards) iff $\underset{\rightarrow}{\lim}(A) \subseteq A$ ($\underset{\leftarrow}{\lim}(A) \subseteq A$), and closed (or closed in $\hat{\Omega}$) iff $\lim(A) \subseteq A$. Now we have

Proposition 3.1 [5]. *Let Ω be a dense unbounded chain.*

(a) *Let $A_i \subseteq \hat{\Omega}$ $(i \in I)$ be closed upwards and $A = \bigcap_{i \in I} A_i$. Then:*

(1) *A is closed upwards.*

(2) *If I is countable and $a \in \bar{\Omega} \cup \{\infty\}$ satisfies $\mathrm{cof}(a) \neq \aleph_0$ and $a \in \underset{\rightarrow}{\lim}(A_i)$ for each $i \in I$, then $a \in \underset{\rightarrow}{\lim}(A)$.*

(b) *Let $\mathrm{cof}(\Omega) \neq \aleph_0$ and $B \subseteq \bar{\Omega}$ be unbounded above and well-ordered. Then $A = \lim(B) \cap \bar{\Omega} \subseteq B$ is closed in $\bar{\Omega}$, unbounded above, and well-ordered with $\mathrm{cof}(a) = \aleph_0$ for each $a \in A \setminus \lim(A)$.*

Notation. Let again Ω be a dense unbounded chain. Each $\alpha \in A(\Omega)$ extends naturally to an isomorphism of $\hat{\Omega}$ which we will also denote by α; let

$F(\alpha)=\{x\in\hat{\Omega}\mid x^{\alpha}=x\}$, the fixed point set of α. We put $F(\Omega)=\{F(\alpha)\mid \alpha\in A(\Omega)\}$. Then we have

$$R(\Omega)=\{\alpha\in A(\Omega)\mid \bar{\Omega}\backslash F(\alpha) \text{ is bounded below in } \bar{\Omega}\}$$

and

$$L(\Omega)=\{\alpha\in A(\Omega)\mid \bar{\Omega}\backslash F(\alpha) \text{ is bounded above in } \bar{\Omega}\},$$

thus each $\alpha\in R(\Omega)$ ($\alpha\in L(\Omega)$) lives "on the right" ("left") in $\bar{\Omega}$, respectively. We let id denote the identity map of Ω.

PROPOSITION 3.2 [5]. *Let Ω be a doubly homogeneous chain. Then:*

(a) *Ω is dense and unbounded. Moreover, the sets* $\{x\in\bar{\Omega}\mid \operatorname{cof}(x)=\aleph_0\}$ and $\{x\in\bar{\Omega}\mid \operatorname{coi}(x)=\aleph_0\}$ *are dense in* $\bar{\Omega}$.

(b) *For any set $A\subseteq\hat{\Omega}$ the following are equivalent:*

(1) $A\in F(\Omega)$;

(2) *$\hat{\Omega}\backslash A$ is a disjoint union of open intervals each with countable coterminality;*

(3) *$-\infty,\infty\in A$, A is closed, and whenever $a\in A$ and* $\operatorname{cof}(a)\neq\aleph_0$ ($\operatorname{coi}(a)\neq\aleph_0$), *then* $a\in\underrightarrow{\lim}(A)$ ($a\in\underleftarrow{\lim}(A)$), *respectively.*

(c) *$F(\Omega)$ is closed under countable intersections.*

(d) *If $\alpha\in A(\Omega)\backslash\{\mathrm{id}\}$, there are $\alpha_1\in L(\Omega)\backslash\{\mathrm{id}\}$, $\alpha_2\in R(\Omega)\backslash\{\mathrm{id}\}$ with $\alpha=\alpha_1\cdot\alpha_2$ and $(\alpha)=(\alpha_1)\cdot(\alpha_2)$.*

(e) $(N_1(A(\Omega)),\subseteq)\cong(N_1(L(\Omega)),\subseteq)\times(N_1(R(\Omega)),\subseteq)$.

Here (c) follows immediately from (b) and (3.1)(a), and the proof of (e) is straightforward by using (d), which is well-known, the distributivity of $N(A(\Omega))$, and (2.13).

NOTATION. For any set $A\subseteq\hat{\Omega}$ we put

$$S(A)=\{a\in A\mid \forall b,c\in\bar{\Omega}: b<a\Rightarrow[b,a]\nsubseteq A, a<c\Rightarrow[a,c]\nsubseteq A\},$$

$$R(A)=\{a\in A\mid \forall b\in\bar{\Omega}: b<a\Rightarrow[b,a]\nsubseteq A;\exists c\in\bar{\Omega}: a<c,[a,c]\subseteq A\},$$

$$L(A)=\{a\in A\mid \exists b\in\bar{\Omega}: b<a,[b,a]\subseteq A;\forall c\in\bar{\Omega}: a<c\Rightarrow[a,c]\nsubseteq A\},$$

$$I(A)=\{a\in A\mid \exists b,c\in\bar{\Omega}: b<a<c,[b,c]\subseteq A\};$$

then $A=S(A)\dot{\cup}R(A)\dot{\cup}L(A)\dot{\cup}I(A)$. If $A=F(\alpha)$ with $\alpha\in A(\Omega)$, we also write $S(\alpha)=S(A)$, $R(\alpha)=R(A)$, $L(\alpha)=L(A)$, $I(\alpha)=I(A)$. For $\alpha\in A(\Omega)$ we always have $-\infty\in R(\alpha)\cup S(\alpha)$ and $\infty\in L(\alpha)\cup S(\alpha)$, and, moreover, $\alpha\in L(\Omega)$ ($\alpha\in R(\Omega)$, $\alpha\in B(\Omega)$, $\alpha=\mathrm{id}$) iff $\infty\in L(\alpha)$ ($-\infty\in R(\alpha)$, $\infty\in L(\alpha)$ and $-\infty\in R(\alpha)$, $F(\alpha)=\hat{\Omega}$), respectively.

PROPOSITION 3.3 [5]. *Let Ω be a dense unbounded chain and $\alpha \in A(\Omega)$. Then $R(\alpha) \cup S(\alpha)$ is closed upwards. If $a \in R(\alpha) \cup S(\alpha)$, $a \neq -\infty$, and* $\operatorname{cof}(a) \neq \aleph_0$, *then $a \in \underrightarrow{\lim}(R(\alpha) \cup S(\alpha))$. The symmetric assertions hold for $L(\alpha) \cup S(\alpha)$.*

The following theorem characterizes the structure of $(N_1(A(\Omega)), \subseteq)$ by that of $(\bar{\Omega}, \leqq)$.

THEOREM 3.4 ([5; Satz 6.18]). *Let Ω be a doubly homogeneous chain and $\alpha, \beta \in A(\Omega)$. The following are equivalent*:

(1) $(\alpha) \subseteq (\beta)$.

(2) *There exists $F \in F(\Omega)$ such that the following two conditions are satisfied*:

(i) $S(\alpha) \cap F \subseteq S(\beta)$;

(ii) *whenever $a, b \in F$ satisfy $a < b$ and $[a, b] \subseteq F(\beta)$, then $[a, b] \subseteq F(\alpha)$.*

Here it was an open question in [5; p. 124] whether under the additional assumption that $\beta \neq \mathrm{id}$ the above theorem can be simplified by leaving condition (2ii) out. We will obtain a negative answer to this problem (see Corollary 3.16).

For the rest of this section let Ω always be a doubly homogeneous chain and the trees $T_r(\Omega)$, $T_l(\Omega)$ constructed according to (2.16). We now start our examination of the structure of $(N_1(A(\Omega)), \subseteq)$.

NOTATION 3.5. For $T = T_r(\Omega)$ or $T = T_l(\Omega)$ and $A \subseteq \bar{\Omega}$, let

$$A_T = ((T^{\sim} \cap S(A)) \cup \{a \in \mathrm{Mid}(T) \mid a_1, a_3 \in A, [a_1, a_3] \not\subseteq A\}) \\ \cap \{a \in T \mid \{x \in T \mid x <_T a\} \subseteq S(A)\}.$$

We also write $A_r = A_{T_r}$, $A_l = A_{T_l}$.

Our first goal is to prove the following structure theorem which establishes a relation between the structures of $(N_1(A(\Omega)), \subseteq)$ and $(T_l^*, \leqq)$, $(T_r^*, \leqq)$ where $T_l = T_l(\Omega)$, $T_r = T_r(\Omega)$.

THEOREM 3.6. *Let Ω be a doubly homogeneous chain and $T_l = T_l(\Omega)$, $T_r = T_r(\Omega)$. Then for $\alpha, \beta \in A(\Omega) \backslash \{\mathrm{id}\}$ the following are equivalent*:

(1) $(\alpha) \subseteq (\beta)$.

(2) *There are $A \in B(T_l)$, $B \in B(T_r)$ such that $(F(\alpha))_l \cap A \subseteq (F(\beta))_l$ and $(F(\alpha))_r \cap B \subseteq (F(\beta))_r$.*

For the proof of this theorem we need some auxiliary results to which we now turn.

LEMMA 3.7. *Let $T = T_r(\Omega)$, $I = I_r$ or $T = T_l(\Omega)$, $I = I_l$. Assume that $A \subseteq T^{\sim}$ satisfies*

(i) *conditions* (i), (ii) *of Definition* 2.2;

(ii) $T^{\sim} \cap M_P \subseteq A$ *for any path* $P \subseteq \bigcup_{j<i} T_j$ $(i \in L(I))$ *with* $P \subseteq A$.

Then A is closed in $\hat{\Omega}$. Hence, in particular, $T^{\sim}$ is closed in $\hat{\Omega}$.

PROOF. Let $\emptyset \neq B \subseteq A$ and $a \in \hat{\Omega}$ with $a = \sup B \notin B$. We claim $a \in A$. For each $x \in B$ let $B_x = \{y \in B \mid x \leqq y\}$. If the following condition

(+) There exist elements $i \in I$, $b \in Z_i$, $v \in B$ such that $B_v \subseteq V_b$ and for all $x \in B_v$ there are $y \in M_b$, $z \in B$ with $x \leqq y \leqq z$

holds, then for all x, y, z as in (+) by (i) there is $y' \in M_b$ with $y \leqq y'$ and $y' \leqq_T z$, hence $x \leqq y' \in A \cap M_b$, showing $a \in \varinjlim(A \cap M_b) \subseteq A$.

Now assume (+) is not satisfied. If $i \in I$, $b \in Z_i$, $v \in B$ and $B_v \subseteq V_b$, let $x \in B_v \setminus M_b$ such that $[x, z] \cap M_b = \emptyset$ for all $z \in B_x$. Since $\min(M_b) < x < b$ $(b < x)$ if $b' < b$ $(b < b')$, there exists $m = \sup\{y \in M_b \mid y < x\} \in M_b$ and $m < x$. If M_b is inversely well-ordered, then $m \notin \varprojlim(M_b)$ and $m < x < \max(M_b)$. Thus, if $m^+ = \min\{y \in M_b \mid m < y\}$, we have $B_x \subseteq (m, m^+)$. Thus $B_x \subseteq V_m$ $(B_x \subseteq V_{m^+})$ if $b' < b$ $(b < b')$. This shows that by transfinite induction we can find a path $P \subseteq \bigcup_{j<i} T_j$ $(i \in L(I))$ with $P \subseteq B \subseteq \bigcup_{j<i} T_j$. Hence $a = \sup P$ and $a = a_P \in M_P \cap T^{\sim}$ by $a \notin B$. Thus $a \in A$ by (ii).

A symmetry-argument shows that A is closed in $\hat{\Omega}$.

The following remark will be used quite often.

REMARK 3.8. Let $\alpha \in A(\Omega)$, $T = T_r(\Omega)$ and $a \in T$. Assume that $X = \{x \in T \mid x <_T a\}$ is non-empty, contains no maximal element, and satisfies $X \subseteq S(\alpha)$. Then $\min(M_X) \in R(\alpha) \cup S(\alpha)$ and $\max(M_X) \in L(\alpha) \cup S(\alpha)$. In particular, $a_1 \in R(\alpha) \cup S(\alpha)$ and $a_3 \in L(\alpha) \cup S(\alpha)$ if $a \in \mathrm{Mid}(T)$.

PROOF. According to our Construction 2.16, we have $\min(M_X) \in \varinjlim(X) \subseteq \varinjlim(S(\alpha)) \subseteq R(\alpha) \cup S(\alpha)$ using (3.3). Similarly for $\max(M_X)$. The last assertion follows immediately from $a_1 = \min(M_X)$, $a_3 = \max(M_X)$ if $a \in \mathrm{Mid}(T)$ as mentioned in the remarks after Construction 2.16.

LEMMA 3.9. *Let $F \in F(\Omega)$ and $U = F \cap T_r^{\sim}$. Then $U_r \in B(T_r)$.*

PROOF. Observe $\infty \in U \subseteq T_r^{\sim}$. We show that U_r satisfies conditions (i)–(iii) of Definition 2.2. Here (i) is trivial. If $a \in U_r$ with $M_a \neq \emptyset$, then $a \in Z_i \cap F$ for some $i \in I_r$ and $U_r \cap M_a = F \cap M_a \in B_a$, hence (ii) holds. Now let $a \in T_r$, $P = \{x \in T_r \mid x <_T a\} \subseteq U_r$ and $P \neq \emptyset$ contain no maximal element. Then either $M_P = \{a_P\}$ or $M_P = \{a_P, b_P, c_P\}$ with $a_p < b_P < c_P$. Hence $a_P \in \lim(P) \subseteq \lim(F) \subseteq F$ and $a_P \in U_r$, since $U = S(U)$. If $|M_P| = 3$, similarly $c_P \in U_r$ and then $b_P \in U_r$

since $[a_P, c_P] \nsubseteq U$. Thus $M_P \subseteq U_r$ in all cases which shows (iii). Hence $U_r \in B(T_r)$.

The following lemma is our main tool for establishing half of the equivalence stated in Theorem 3.6.

LEMMA 3.10. *Let $\alpha, \beta \in R(\Omega)\backslash\{\mathrm{id}\}$. If there exists a $B \in B(T_r)$ with $(F(\alpha))_r \cap B \subseteq (F(\beta))_r$, then $\alpha \in (\beta)$.*

PROOF. Let $B \in B(T_r)$ satisfy $(F(\alpha))_r \cap B \subseteq (F(\beta))_r$ and w.l.o.g. also conditions (i)–(iii) of Definition 2.2. We will construct an $F \in F(\Omega)$ satisfying conditions (2i, ii) of Theorem 3.4; then $\alpha \in (\beta)$ by this theorem.

We abbreviate $(T, \leqq_T) = (T_r, \leqq_r)$ and $I = I_r$. For each

$$a \in S(\alpha) \cap S(\beta) \cap B \cap Z_i, \qquad i \in I,$$

let

$$C_a = M_a \cap B \cap (R(\alpha) \cup S(\alpha)) \cap (R(\beta) \cup S(\beta)) \qquad \text{if } a' < a,$$

and

$$C_a = M_a \cap B \cap (L(\alpha) \cup S(\alpha)) \cap (L(\beta) \cup S(\beta)) \qquad \text{if } a < a'.$$

Since $a \in S(\alpha) \cap S(\beta)$ and B satisfies condition (2.2)(ii), C_a is closed and unbounded above (below) in M_a by (3.1)(a) and (3.3). Let $U_a = (\lim(C_a))\backslash\{a\} \subseteq C_a$. Then again $U_a \in B_a$ by (3.1)(b), and for later use note that if $a' < a$ ($a < a'$), then for every $w \in U_a \cup \{a\}$ with $\mathrm{cof}(w) \neq \aleph_0$ ($\mathrm{coi}(a) \neq \aleph_0$) we have $w \in \underrightarrow{\lim}(U_a)$ ($w \in \underleftarrow{\lim}(U_a)$).

Now we define a set $W \subseteq T^{\sim}$ by transfinite induction. Let $W_0 = \{\infty\}$. If $W_i \subseteq T_i$ is already defined for some $i \in I$, let

$$V_{i+1} = \dot{\bigcup} \{U_a \mid a \in W_i \cap Z_i \cap S(\alpha) \cap S(\beta)\} \quad \text{and} \quad W_{i+1} = W_i \dot{\cup} V_{i+1} \subseteq T_{i+1}.$$

If $i \in L(I)$ and $W_j \subseteq T_j$ are already defined for $j < i$, let

$$V_i = \dot{\bigcup} \{T^{\sim} \cap M_P \mid P \subseteq \bigcup_{j<i} T_j \text{ path with } P \subseteq \bigcup_{j<i} W_j\} \quad \text{and} \quad W_i = \bigcup_{j<i} W_j \dot{\cup} V_i.$$

We put $W = \bigcup_{i \in I} W_i$.

Step 1. We establish several properties of W.

(I) If $w \in W$, $x \in T^{\sim}$ and $x <_T w$, then $x \in W \cap S(\alpha) \cap S(\beta)$.

(II) If $x \in T^{\sim}\backslash S(\alpha)$ and $x' < x$ ($x < x'$), then $[x', x) \cap W = \varnothing$ ($(x, x'] \cap W = \varnothing$).

This follows immediately from (I), since (e.g.) any $w \in [x', x) \cap W$ satisfies $x <_T w$.

(III) W is closed in $\bar{\Omega}$.

We apply Lemma 3.7. If $a \in W \cap Z_i \cap S(\alpha) \cap S(\beta)$ for some $i \in I$, then $U_a \in B_a$, hence (3.7)(i) holds. If $P \subseteq \bigcup_{j<i} T_j$ $(i \in L(I))$ is a path with $P \subseteq W$, then $P \subseteq \bigcup_{j<i} W_j$ and $T^{\sim} \cap M_P \subseteq W_i \subseteq W$, which shows (3.7)(ii).

(IV) $W \subseteq B \cap (R(\alpha) \cup L(\alpha) \cup S(\alpha))$. Moreover, if $w \in W \cap S(\alpha)$, then $w \in S(\beta)$, and if $w \in W \cap R(\alpha)$ $(w \in W \cap L(\alpha))$, then $w < w'$ $(w' < w)$ and $w \in R(\beta) \cup S(\beta)$ $(w \in L(\beta) \cup S(\beta))$.

Let $w \in W$. If $w \in U_a$ for some $a \in W_i \cap Z_i \cap S(\alpha) \cap S(\beta)$, $i \in I$, then $w \in B \cap (R(\alpha) \cup L(\alpha) \cup S(\alpha))$ by definition; moreover, if $w \in R(\alpha)$, we get $a' < a$, hence $w < w'$, and $w \in R(\beta) \cup S(\beta)$. Now let $w \in T^{\sim} \cap M_P$ for some path $P \subseteq \bigcup_{j<i} T_{j+1}$ $(i \in L(I))$ with $P \subseteq \bigcup_{j<i} W_{j+1}$. Then $P \subseteq B$, hence $w \in B$, since B satisfies conditions (2.2)(i, iii). Also $P \subseteq \{x \in T \mid x <_T w\} \subseteq S(\alpha) \cap S(\beta)$ by (I). Hence either $w = \min(M_P) \in (R(\alpha) \cup S(\alpha)) \cap (R(\beta) \cup S(\beta))$ or $w = \max(M_P) \in (L(\alpha) \cup S(\alpha)) \cap (L(\beta) \cup S(\beta))$ by (3.8). Consequently, $w \in R(\alpha)$ now implies $w \in R(\beta) \cup S(\beta)$ and $w \in \overrightarrow{\lim}(P) < \overleftarrow{\lim}(P)$, hence $w < w'$ by Construction 2.16. Similarly for $w \in W \cap L(\alpha)$. If $w \in W \cap S(\alpha)$, then $\{x \in T \mid x <_T a\} \subseteq S(\alpha) \cap S(\beta)$ by (I) and $w \in B \cap T^{\sim}$ as previously shown, hence $w \in (F(\alpha))_r \cap B \subseteq (F(\beta))_r$ and thus $w \in S(\beta)$.

(V) Let $w \in W$ and $\operatorname{cof}(w) \neq \aleph_0$ $(\operatorname{coi}(w) \neq \aleph_0)$. If $w \in R(\alpha) \cup S(\alpha)$ $(w \in L(\alpha) \cup S(\alpha))$, then $w \in \overrightarrow{\lim}(W)$ $(w \in \overleftarrow{\lim}(W))$.

W.l.o.g. assume $\operatorname{cof}(w) \neq \aleph_0$ and $w \in R(\alpha) \cup S(\alpha)$. If $w' < w$, then $w \in R(\alpha)$ by (IV), hence (using (IV) again) $w \in S(\alpha) \cap W \cap Z_i \subseteq S(\beta)$ for some $i \in I$, thus $U_w \subseteq W$ and $w \in \overrightarrow{\lim}(U_w)$. Now assume $w \leqq w'$. If $w \in U_a$ for some $a \in W \cap Z_i \cap S(\alpha) \cap S(\beta)$, then $a' < a$, $a \in B$ by (IV), and again $w \in \overrightarrow{\lim}(U_a)$ as previously mentioned. Finally, let $w \in T^{\sim} \cap M_P$ for some path $P \subseteq \bigcup_{j<i} T_j$ $(i \in L(I))$ with $P \subseteq W$. Now $w < w'$ implies $w = a_p \in \overrightarrow{\lim}(P) \subseteq \overrightarrow{\lim}(W)$. Hence $w \in \overrightarrow{\lim}(W)$ in all cases.

Step 2. Definition of $F \in F(\Omega)$.

According to (IV) and (3.2)(a), for each $w \in W \cap (R(\alpha) \cup L(\alpha))$ we can now choose an element $\bar{w} \in \bar{\Omega}$ satisfying

(a) $w < \bar{w} < w'$, $[w, \bar{w}] \subseteq F(\alpha)$, and $\operatorname{coi}(\bar{w}) = \aleph_0$, if $w \in R(\alpha)$,

(b) $w' < \bar{w} < w$, $[\bar{w}, w] \subseteq F(\alpha)$, and $\operatorname{cof}(\bar{w}) = \aleph_0$, if $w \in L(\alpha)$,

(c) $[-\infty, \bar{w}] \nsubseteq F(\beta)$ if $w = \infty \in L(\alpha)$;

moreover, if $w_1 \in W \cap R(\alpha)$, $w_2 \in W \cap L(\alpha)$ (hence $w_1, w_2 \in F(\beta)$ by (IV)), and $w_1 = a_P < c_P = w_2$ for some path $P \subseteq \bigcup_{j<i} T_j$ $(i \in L(I))$ with $P \subseteq W$, the

choice of $\bar{w}_1, \bar{w}_2$ can be done such that, in addition to the other properties, also $[\bar{w}_1, \bar{w}_2] \not\subseteq F(\beta)$ holds if $[w_1, w_2] \not\subseteq F(\alpha)$.

Also, let $z \in \bar{\Omega}$ satisfy $z < \infty'$, $[-\infty, z] \subseteq F(\alpha) \cap F(\beta)$, and $\mathrm{coi}(z) = \aleph_0$. Now let

$$F = [-\infty, z] \cup W \cup \bigcup_{w \in W \cap R(\alpha)} [w, \bar{w}] \cup \bigcup_{w \in W \cap L(\alpha)} [\bar{w}, w].$$

We claim that F is closed in $\hat{\Omega}$. Assume $X \subseteq F$ and $x = \sup X \notin X$; we want to show $x \in F$. W.l.o.g. let $X \cap [-\infty, z] = \emptyset$. In case there are $w \in W \cap R(\alpha)$ ($w \in W \cap L(\alpha)$), $a \in X$ such that $b \in (w, w')$ ($b \in (w', w)$) for all $b \in X$ with $a \leqq b$, then $W \cap (w, w') = \emptyset$ ($W \cap (w', w) = \emptyset$) by (II), hence $b \in [w, \bar{w}]$ ($b \in [\bar{w}, w]$) for all $b \in X$ with $a \leqq b$, thus $x \in F$; otherwise for each $b \in X$ there is an element $w \in W$ with $b < w < x$, and so we obtain a set $Y \subseteq W$ with $x \notin Y$ and $x = \sup Y$, thus $x \in W \subseteq F$ by (III). Hence, by a symmetry-argument, F is closed.

According to our construction and to (V), we furthermore have $a \in \overrightarrow{\lim}(F)$ ($a \in \overleftarrow{\lim}(F)$) for each $a \in F$ with $\mathrm{cof}(a) \neq \aleph_0$ ($\mathrm{coi}(a) \neq \aleph_0$). Since $-\infty, \infty \in F$, we conclude $F \in F(\Omega)$ by (3.2)(b).

Step 3. We claim that F satisfies conditions (2i, ii) of Theorem 3.4.

First we obtain $S(\alpha) \cap F = S(\alpha) \cap W \subseteq S(\beta)$ by (IV), as claimed. Now let $x, y \in F$ with $x < y$ and $[x, y] \subseteq F(\beta)$. We will show $[x, y] \subseteq F(\alpha)$. If $x, y \in [-\infty, z]$, we immediately get $[x, y] \subseteq F(\alpha)$. On the other hand, the assumption $x \leqq z < y$ leads to a contradiction in all cases: If $\infty \in L(\alpha)$, we have $z < \infty' < \bar{\infty} \leqq y$, $[-\infty, z] \subseteq F(\beta)$ and thus $[z, \bar{\infty}] \not\subseteq F(\beta)$ by (c); if $\infty \in S(\alpha)$ and $\mathrm{cof}(\Omega) = \aleph_0$, we obtain $W = \{\infty\}$, $F = [-\infty, z] \cup \{\infty\}$, $y = \infty \in S(\alpha) \cap F \subseteq S(\beta)$, hence $[x, y] \not\subseteq F(\beta)$; if $\infty \in S(\alpha)$ and $\mathrm{cof}(\Omega) \neq \aleph_0$, we have $z < m = \min(U_\infty) \in R(\beta) \cup S(\beta)$, thus $[z, m] \not\subseteq F(\beta)$, $m \leqq y$ and hence again $[x, y] \not\subseteq F(\beta)$.

Therefore we can now assume $x, y \notin [-\infty, z]$ and, furthermore, $x, y \notin W \cap S(\alpha)$, since $W \cap S(\alpha) \subseteq S(\beta)$. Thus by (IV) there are $w_1, w_2 \in W \cap (R(\alpha) \cup L(\alpha))$ such that $x \in W_1$, $y \in W_2$ if we put $W_i = [w_i, \bar{w}_i]$ ($W_i = [\bar{w}_i, w_i]$) if $w_i \in R(\alpha)$ ($w_i \in L(\alpha)$), for $i = 1, 2$. By (II), we obtain either $W_1 = W_2$ or $W_1 \cap W_2 = \emptyset$. If $W_1 = W_2$, immediately $[x, y] \subseteq W_1 \subseteq F(\alpha)$ by construction. So now let us assume $W_1 \neq W_2$.

Since $W_1 \cap W_2 = \emptyset$ and $x < y$, we have $W_1 < W_2$. In particular, this shows $[\max\{w_1, \bar{w}_1\}, \min\{w_2, \bar{w}_2\}] \subseteq F(\beta)$. This yields a contradiction if $\max\{w_1, \bar{w}_1\} = w_1$, since then $w_1 \in W \cap L(\alpha) \subseteq L(\beta) \cup S(\beta)$ by (IV). Hence $w_1 < \bar{w}_1$, $w_1 \in R(\alpha)$, and similarly $\bar{w}_2 < w_2$, $w_2 \in L(\alpha)$.

Assume we had $w_1 \in U_a$ for some $a \in W \cap Z_i \cap S(\alpha) \cap S(\beta)$, $i \in I$. Then $a' < a$ by $w_1 \in R(\alpha)$, thus M_a and U_a are well-ordered. Let $w = \min\{y \in U_a \mid w_1 < y\}$, then $w_1 < \bar{w}_1 < w_1' < w < w_2$. But $w \notin [w_2', w_2)$ by (II), so $w < w_2' < \bar{w}_2 < w_2$ and $[\bar{w}_1, w] \subseteq [x, y] \subseteq F(\beta)$. This contradicts $w \in U_a \subseteq R(\beta) \cup S(\beta)$.

This and a symmetrical argument shows that $w_i \in T^{\sim} \cap M_{P_i}$ for some paths $P_i \subseteq \bigcup_{j<k_i} T_j$ $(k_i \in L(I))$ with $P_i \subseteq W$, $i = 1,2$. Since $w_1 < \bar{w}_1 < w_1'$, $w_2' < w_2$, we get $|M_{P_i}| = 3$ $(i = 1,2)$ and $w_1 = a_{P_1}$, $w_2 = c_{P_2}$. We put $a_i = a_{P_i}$, $c_i = c_{P_i}$ $(i = 1,2)$ for abbreviation. Hence $x \in W_1 \subseteq [a_1, c_1]$ and $y \in W_2 \subseteq [a_2, c_2]$. We claim $M_{P_1} = M_{P_2}$.

Assume $M_{P_1} \neq M_{P_2}$. Since $w_i \notin S(\alpha)$, $T^{\sim} \cap M_{P_i} \subseteq W$ $(i = 1,2)$, we do not have $c_2 <_T a_1$ or $a_1 <_T c_2$ by (I). Next observe that each $x \in T^{\sim}$ with $x <_T c_1$ satisfies $x < a_1 = w_1$, hence $x \in S(\beta)$ by (I). Consequently $c_1 = \max(M_{P_1}) \in L(\beta) \cup S(\beta)$ by (3.8); similarly $a_2 \in R(\beta) \cup S(\beta)$. Now we distinguish between four cases.

Case 1. Assume $a_1' < a_2$. Then $[x, a_2] \subseteq [x, y] \subseteq F(\beta)$ contradicting $a_2 \in R(\beta) \cup S(\beta)$.

Case 2. Let $a_2 \in (a_1, a_1')$. Then $a_1 <_T \{a_2, c_2\}$, a contradiction as already mentioned.

Case 3. Let $a_2 < a_1$ and $c_2 < a_1$. This contradicts $x < y$.

Case 4. Assume $a_2 < a_1 < c_1 < c_2$. Then $c_2' < a_1$ implies $c_2 <_T a_1$, a contradiction. Consequently $a_1 < c_1 < c_2' = w_2' < y$. But now $[c_1, y] \subseteq [x, y] \subseteq F(\beta)$, a final contradiction to $c_1 \in L(\beta) \cup S(\beta)$.

Thus $M_{P_1} = M_{P_2}$, $w_1 = a_1 = a_2$, $w_2 = c_2 = c_1$. Since $[\bar{w}_1, \bar{w}_2] \subseteq [x, y] \subseteq F(\beta)$, from our construction of F it now follows that $[w_1, w_2] \subseteq F(\beta)$, thus $b_{P_1} \notin (F(\beta))_r$. But $w_1, w_2 \in W \subseteq F(\alpha) \cap B$, so $b_{P_1} \in B$ since B satisfies conditions (2.2)(i, iii). Hence $b_{P_1} \notin (F(\alpha))_r$. Since $\{z \in T \mid z <_T w_1\} \subseteq S(\alpha)$ by (I), we obtain $[x, y] \subseteq [a_1, c_1] \subseteq F(\alpha)$. Hence $[x, y] \subseteq F(\alpha)$ in all cases, and (3.4)(2) is proved.

Now we can come to the

Proof of Theorem 3.6. (1)$\rightarrow$(2). According to Theorem 3.4, there exists $F \in F(\Omega)$ satisfying conditions (2i, ii) of (3.4). Let $A = (F(\beta) \cap F \cap T_l^{\sim})_l$, $B = (F(\beta) \cap F \cap T_r^{\sim})_r$. By Lemma 3.9 we have $B \in B(T_r)$. We claim $(F(\alpha))_r \cap B \subseteq (F(\beta))_r$. Let $a \in (F(\alpha))_r \cap B$. Then $\{x \in T_r \mid x <_r a\} \subseteq S(\alpha) \cap F \subseteq S(\beta)$ by condition (2i) of (3.4). If $a \in T_r^{\sim}$, then $a \in S(\alpha) \cap F \subseteq S(\beta)$. If $a \in \mathrm{Mid}(T_r)$, then $a_1, a_3 \in F(\alpha) \cap (F(\beta) \cap F)$ and $[a_1, a_3] \nsubseteq F(\alpha)$, hence $[a_1, a_3] \nsubseteq F(\beta)$ by condition (2ii) of (3.4), thus $a \in (F(\beta))_r$. This shows the assertion for B. By a symmetrical argument, it follows also for A.

(2)→(1). By (3.2)(d), there are $\alpha_1, \beta_1 \in L(\Omega)\backslash\{\mathrm{id}\}$, $\alpha_2, \beta_2 \in R(\Omega)\backslash\{\mathrm{id}\}$ with $\alpha = \alpha_1 \cdot \alpha_2$, $\beta = \beta_1 \cdot \beta_2$, $(\alpha) = (\alpha_1) \cdot (\alpha_2)$, $(\beta) = (\beta_1) \cdot (\beta_2)$. We claim $(\alpha_i) \subseteq (\beta_i)$ for $i = 1, 2$. If $\mathrm{cof}(\Omega) \neq \aleph_0$ ($\mathrm{cof}(\Omega) = \aleph_0$), let $y \in \bar{\Omega}$ and $z \in M_\infty$ ($z \in \bar{\Omega}$) satisfy $y < z$ and $x^\alpha = x^{\alpha_2}$, $x^\beta = x^{\beta_2}$ for all $x \in \bar{\Omega}$ with $y \leqq x$. Let $C = [z, \infty] \cap T_r$. Obviously, $C \in B(T_r)$ and $(F(\alpha))_r \cap C = (F(\alpha_2))_r \cap C$, $(F(\beta))_r \cap C = (F(\beta_2))_r \cap C$. Thus $(F(\alpha_2))_r \cap B \cap C \subseteq (F(\beta_2))_r$ by (2), hence $(\alpha_2) \subseteq (\beta_2)$ by Lemma 3.10. By symmetry, we obtain $(\alpha_1) \subseteq (\beta_1)$. Hence $(\alpha) \subseteq (\beta)$ as claimed.

Next we want to use Theorem 3.6 to establish an isomorphism from $(N_1(A(\Omega)), \subseteq)$ onto $(T_l^*, \leqq) \times (T_r^*, \leqq)$, where $T_l = T_l(\Omega)$, $T_r = T_r(\Omega)$ as usual. For this we will now derive in (3.11)–(3.15) some properties of sets $A \subseteq \hat{\Omega}$ which are of the form $A = (F(\alpha))_r$ for some $\alpha \in A(\Omega)$. First we show:

Lemma 3.11. *Let $T = T_r(\Omega)$. Then $(F(\alpha))_r \in T^+$ for any $\alpha \in A(\Omega)$.*

Proof. Let $\alpha \in A(\Omega)$ and $A = (F(\alpha))_r$. Condition (2.3)(i) holds trivially. Now let $a \in T$ such that $X = \{x \in T \mid x <_T a\}$ contains no maximal element. If $a \in \mathrm{Mid}(T)$ and $A \cap \{a_1, a_3\} \neq \varnothing$, then $X \subseteq S(\alpha)$ and $a_1, a_3 \in F(\alpha)$ by (3.8). Since $a_1 \in S(\alpha)$ or $a_3 \in S(\alpha)$, we have $[a_1, a_3] \not\subseteq F(\alpha)$, in total $a \in A$. This shows (2.3)(ii). If, on the other hand, $M_X = \{a\}$ and $X \subseteq A$, then $a \in (R(\alpha) \cup S(\alpha)) \cap (L(\alpha) \cup S(\alpha)) = S(\alpha)$ by (3.8), hence $a \in A$, and thus (2.3)(iii) holds.

Next we want to prove the converse to (3.11), namely that each set $A \in T^+$ (where $T = T_r(\Omega)$) is of the form $A = (F(\alpha))_r$ for some $\alpha \in R(\Omega)$. Here we use a result from [5] which characterizes all sets $A \subseteq \hat{\Omega}$ of the form $A = S(\alpha)$ for some $\alpha \in A(\Omega)$.

Definition 3.12 [5; Definition 7.1]. A set $A \subseteq \hat{\Omega}$ is closed to the interior in $\hat{\Omega}$ if it satisfies the following conditions:

(i) If $\infty \in \underrightarrow{\lim}(A)$ ($-\infty \in \underleftarrow{\lim}(A)$), then $\infty \in A$ ($-\infty \in A$).

(ii) If $x \in \bar{\Omega}$ and $x \in \underrightarrow{\lim}(A) \cap \underleftarrow{\lim}(A)$, then $x \in A$.

Lemma 3.13 [5; Lemma 7.3]. (a) *If $\alpha \in A(\Omega)$, then $I(S(\alpha)) = \varnothing$ and $S(\alpha)$ is closed to the interior in $\hat{\Omega}$.*

(b) *Let $A \subseteq \hat{\Omega}$ be closed to the interior in $\hat{\Omega}$ and $I(A) = \varnothing$. Then there exists an $\alpha \in A(\Omega)\backslash\{\mathrm{id}\}$ with $S(\alpha) = A$ such that $[a, b] \not\subseteq F(\alpha)$ whenever $a, b \in F(\alpha)$, $a < b$, and $[a, b]$ is a maximal interval in $\hat{\Omega}$ with $[a, b] \cap S(\alpha) = \varnothing$.*

Lemma 3.14. *Let $T = T_r(\Omega)$ and $A \in T^+$. Then $A \cap T^-$ is closed to the interior in $\hat{\Omega}$.*

PROOF. Let $A \in T^+$ and $A^- = A \cap T^-$. Obviously, $-\infty \notin \overleftarrow{\lim}(A^-)$. If $\infty \in \overrightarrow{\lim}(A^-)$, then $\mathrm{cof}(\Omega) \neq \aleph_0$, $\infty \leqq_T A^-$ and $\infty \in A^-$. Now let $x \in \bar{\Omega}$ satisfy $x \in \overrightarrow{\lim}(A^-) \cap \overleftarrow{\lim}(A^-)$. Then $x \in T^-$, since T^- is closed in $\hat{\Omega}$ by Lemma 3.7. If $x \neq x'$ and w.l.o.g. $x' < x$, there exists $y \in A^-$ with $x' < y < x$, hence $x <_T y$ and $x \in A^-$. Now assume $x = x'$. Let $y \in T$ with $y <_T x$ and w.l.o.g. $y < x$. There is an $a \in A^-$ with $y < a < x$, hence $y <_T a$ and $y \in A$. This shows $\{y \in T \mid y <_T x\} \subseteq A$. Since $A \in T^+$, we obtain $x \in A^-$.

The following lemma contains the announced converse to Lemma 3.11:

LEMMA 3.15. *Let $T = T_r(\Omega)$ and $A \in T^+$. Then there is an $\alpha \in R(\Omega)\backslash\{\mathrm{id}\}$ with $(F(\alpha))_r = A$ and $S(\alpha) = A \cap T^-$.*

PROOF. *Step 1.* Construction of $\alpha \in R(\Omega)$.

According to Lemma 3.14, $A^- = A \cap T^-$ is closed to the interior in $\hat{\Omega}$. Since $I(A^-) = I(T) = \emptyset$, by Lemma 3.13(b) there is an element $\beta \in A(\Omega)\backslash\{\mathrm{id}\}$ with $S(\beta) = A^-$ such that $[a, b] \nsubseteq F(\beta)$ whenever $a, b \in F(\beta)$, $a < b$, and $[a, b]$ is a maximal interval with $[a, b] \cap S(\beta) = \emptyset$. Since $-\infty \notin S(\beta)$, we have $\beta \in R(\Omega)$. Let

$$Z = \{a \in \mathrm{Mid}(T) \mid a \notin A, \{x \in T \mid x <_T a\} \subseteq A^-\}.$$

If $a \in Z$, then $a_1 \in R(\beta) \cup S(\beta)$ by (3.8) and $a_1 \notin S(\beta)$, since otherwise we had $a_1 \in A$ and $a \in A$ by $A \in T^+$, a contradiction. Thus $a \in Z$ implies $a_1 \in R(\beta)$ and, by symmetry, $a_3 \in L(\beta)$. Now put $B = F(\beta) \cup \bigcup_{a \in Z}[a_1, a_3]$. Since $a_1, a_3 \in F(\beta)$ for each $a \in Z$, we get obviously $B \in F(\Omega)$. Let $\alpha \in R(\Omega)$ satisfy $F(\alpha) = B$.

Step 2. We show $S(\alpha) = S(\beta)$.

Clearly, $S(\alpha) \subseteq S(\beta)$. Let $a \in S(\beta) = A \cap T^-$. Then $a \in F(\alpha)$; it suffices to show that $a \in R(\alpha) \cup S(\alpha)$, since then also $a \in L(\alpha) \cup S(\alpha)$ by symmetry, hence $a \in S(\alpha)$. So let $x \in \bar{\Omega}$ with $x < a$; we claim $[x, a) \nsubseteq F(\alpha)$. We first note

(*) If $x \leqq y < z \leqq a$, $z \in A^-$ and $(y, z) \cap A^- = \emptyset$, then $[x, a) \nsubseteq F(\alpha)$.

PROOF OF (*). For any $b \in Z$ we have $b_3 \in \overleftarrow{\lim}(A^-)$, hence $b_3 \notin (y, z)$, and $z \notin [b_1, b_3]$ by $A \in T^+$ and $b_1, b_3 \notin A$. Thus $(y, z) \cap \bigcup_{b \in Z}[b_1, b_3] = \emptyset$. Hence $[x, a) \nsubseteq F(\alpha)$, since otherwise $(y, z) \subseteq F(\alpha)$ and $(y, z) \subseteq F(\beta)$, a contradiction to $z \in S(\beta)$. This proves (*).

We now show $[x, a) \nsubseteq F(\alpha)$. Because of (*), we can assume $a \in \overrightarrow{\lim}(A^-)$. We distinguish between two cases.

Case I. Assume either $(+)$ $a < a'$ and $a \in M_b$ for some $b \in T^{\sim}$,
or $(++)$ $a' < a$.

If $(+)$ holds, we have $a \in \overrightarrow{\lim}(M_b)$ since otherwise there is an element $d \in M_b \dot\cup \{b\}$ with $d' < a$ and $[d', a) \cap T = \emptyset$, a contradiction to $a \in \overrightarrow{\lim}(A^{\sim}) \subseteq \overrightarrow{\lim}(T^{\sim})$. Also $a \in \overrightarrow{\lim}(M_a)$ in case of $(++)$. Hence, for $c = b$ or $c = a$, respectively, we have $a \in \overrightarrow{\lim}(M_c)$ and $y < y'$ for each $y \in M_c$. Choose $y \in M_c$ with $x < y < a$. Since $a \in \overrightarrow{\lim}(A^{\sim})$ and $A \in T^+$, we get $D = M_c \cap (y', a) \cap A^{\sim} \neq \emptyset$. Let $z = \min D$. Then $z \in A^{\sim}$ and $(y', z) \cap A^{\sim} = \emptyset$ by $A \in T^+$ and minimality of z. Now $(*)$ yields $[x, a) \not\subseteq F(\alpha)$.

Case II. Assume $a \leqq a'$ and $a \in M_P$ for some path $P \subseteq \bigcup_{j<i} T_j$ $(i \in L(I_r))$.

Thus $a = a_P = \min(M_P)$ and $X = \{x \in T \mid x <_T a\} \subseteq S(\beta)$ by $A \in T^+$. But then $Y = X \cap \bigcup_{j<i} T_{j+1} \subseteq S(\beta)$ satisfies $Y \subseteq R(\alpha) \cup S(\alpha)$ by Case I. Consequently, $a \in \overrightarrow{\lim}(Y) \subseteq \overrightarrow{\lim}(R(\alpha) \cup S(\alpha)) \subseteq R(\alpha) \cup S(\alpha)$ by (3.3) which proves our claim.

Step 3. We show $\alpha \in R(\Omega) \setminus \{\mathrm{id}\}$ and $(F(\alpha))_r = A$.

If $\infty \in S(\beta)$, then $\infty \in S(\alpha)$ by Step 2, and $\alpha \neq \mathrm{id}$. But if $\infty \notin S(\beta)$, then $A^{\sim} = \emptyset$, $Z = \emptyset$ and $F(\alpha) = B = F(\beta)$ and again $\alpha \neq \mathrm{id}$ by $\beta \neq \mathrm{id}$. Thus $\alpha \in R(\Omega) \setminus \{\mathrm{id}\}$. If $a \in (F(\alpha))_r \cap T^{\sim}$ or $a \in A \cap T^{\sim}$, then $\{x \in T \mid x <_T a\} \subseteq S(\alpha) = S(\beta)$. This shows $A \cap T^{\sim} = S(\beta) = S(\alpha) = (F(\alpha))_r \cap T^{\sim}$, and it remains to prove that $A \cap \mathrm{Mid}(T) = (F(\alpha))_r \cap \mathrm{Mid}(T)$.

So let $a \in \mathrm{Mid}(T)$ and $X = \{x \in T \mid x <_T a\} \subseteq T^{\sim}$. We have to show that $a \in (F(\alpha))_r$ iff $a \in A$. First note that if $X \subseteq A$, by $X \subseteq S(\beta) = S(\alpha)$ and (3.8) we obtain $a_1, a_3 \in F(\alpha) \cap F(\beta)$.

Now if $a \in (F(\alpha))_r$, we have $X \subseteq S(\alpha) \cap T^{\sim} = S(\beta) = A^{\sim}$, $a_1, a_3 \in F(\alpha)$ and $[a_1, a_3] \not\subseteq F(\alpha) = B$. This shows $a \notin Z$ by definition of B, thus $a \in A$.

Conversely, assume $a \in A$ which implies $X \subseteq A$ by $A \in T^+$ and thus $a_1, a_3 \in F(\alpha) \cap F(\beta)$ as mentioned before. We claim $[a_1, a_3] \not\subseteq F(\alpha)$. If $[a_1, a_3] \cap S(\beta) \neq \emptyset$, we use $A \in T^+$ to obtain $\{a_1, a_3\} \cap S(\beta) \neq \emptyset$, hence $a_1 \in S(\alpha)$ or $a_3 \in S(\alpha)$, thus $[a_1, a_3] \not\subseteq F(\alpha)$. Now assume $[a_1, a_3] \cap S(\beta) = \emptyset$. This shows that if $b \in Z$, then $b_1, b_3 \notin (a_1, a_3)$; furthermore, $b \neq a$ since $a \in A$, thus $b_1, b_3 \notin [a_1, a_3]$; also $a \notin [b_1, b_3]$ since $a \in A$, $b_1, b_3 \notin A$. Thus $[b_1, b_3] \cap [a_1, a_3] = \emptyset$ for any $b \in Z$. Hence $[a_1, a_3] \subseteq F(\alpha) = B$ would imply $[a_1, a_3] \subseteq F(\beta)$. But by $X \subseteq A$ we have $a_1 \in \overrightarrow{\lim}(S(\beta))$, $a_3 \in \overleftarrow{\lim}(S(\beta))$, so $[a_1, a_3]$ is a maximal interval with $[a_1, a_3] \cap S(\beta) = \emptyset$. By our initial assumption on β, this implies $[a_1, a_3] \not\subseteq F(\beta)$. Therefore $[a_1, a_3] \not\subseteq F(\alpha)$ also in this case, which shows $a \in (F(\alpha))_r$.

Now we are able to give the

PROOF OF THEOREM 2.9. Because of (3.2)(e) and a symmetry-argument, it suffices to show $(N_1(R(\Omega)), \subseteq) \cong (T_r^*, \leqq)$. Define $\psi: N_1(R(\Omega)) \to T_r^*$ by $(\alpha)^\psi = [(F(\alpha))_r]$, if $\alpha \in R(\Omega)\backslash\{\mathrm{id}\}$. The fact that ψ is well-defined, surjective, and an order-isomorphism follows from Lemma 3.11, Lemma 3.15, and Theorem 3.6 (cf. Lemma 3.10).

As an application of Theorem 3.6, we will now answer a question in [5; p. 124] whether in (3.4) ((1) $\Leftrightarrow$ (2)) condition (2ii) is really necessary. The answer will be positive.

COROLLARY 3.16. *Let Ω be a doubly homogeneous chain and $T_l = T_l(\Omega)$, $T_r = T_r(\Omega)$. Then the following are equivalent*:

(1) *Whenever $\alpha, \beta \in A(\Omega)\backslash\{\mathrm{id}\}$, the existence of an $F \in F(\Omega)$ with $S(\alpha) \cap F \subseteq S(\beta)$ implies $\alpha \in (\beta)$.*

(2) *Whenever $T = T_l$, $I = I_l$, or $T = T_r$, $I = I_r$, then either $|I| < \aleph_0$ or $\omega \in I$ and $T_\omega \cap \mathrm{Mid}(T) \cap B = \varnothing$ for some $B \in B(T)$.*

(3) *There are $A \in B(T_l)$, $B \in B(T_r)$ such that $\mathrm{Mid}(T_l) \cap A = \varnothing$ and $\mathrm{Mid}(T_r) \cap B = \varnothing$.*

PROOF. (1)$\to$(2). W.l.o.g. assume $T = T_r$, $I = I_r$, $\omega \in I$. Let

$$A = T_\omega \backslash \{a_1, a_3 \mid a \in T_\omega \cap \mathrm{Mid}(T)\} \text{ and } B = T_\omega \backslash \{a, a_1, a_3 \mid a \in T_\omega \cap \mathrm{Mid}(T)\}.$$

Then $A, B \in T^+$, hence by Lemma 3.15 there are $\alpha, \beta \in R(\Omega)\backslash\{\mathrm{id}\}$ with $(F(\alpha))_r = A$, $S(\alpha) = A \cap T^-$, $(F(\beta))_r = B$, and $S(\beta) = B \cap T^-$. Thus $S(\alpha) = S(\beta)$, and now (1) implies $\alpha \in (\beta)$. By Theorem 3.6, there exists $C \in B(T)$ with $A \cap C \subseteq B$. This implies

$$T_\omega \cap \mathrm{Mid}(T) \cap C = A \cap T_\omega \cap \mathrm{Mid}(T) \cap C \subseteq B \cap T_\omega \cap \mathrm{Mid}(T) = \varnothing.$$

(2)$\to$(3). We prove the assertion for $T = T_r$, $I = I_r$. W.l.o.g. assume $\omega \in I$ and that $B \in B(T)$ as in (2) satisfies conditions (2.2)(i)–(iii). Put $C = (T_\omega \backslash \mathrm{Mid}(T)) \cap B$. We claim that C also satisfies (2.2)(i)–(iii). Here (i) is trivial. If $x \in C \cap (T_\omega \backslash \bigcup_{j<\omega} T_j)$, there is a path $P \subseteq \bigcup_{j<\omega} T_j$ with $x \in M_P$, hence $M_P \subseteq B \cap T_\omega$ and $|M_P| = 1$ by (2). But since $M_P = \{x\} \subseteq T$, we have $\cot(x) = \aleph_0$, hence $x \notin Z_\omega$. This shows (ii): If $a \in C \cap Z_i$ for $i \in I$, then $i < \omega$ and $M_a \cap C = M_a \cap B \in B_a$. Now (iii) is straight-forward. Thus $C \in B(T)$ and $\mathrm{Mid}(T) \cap C = \varnothing$.

(3)$\to$(1). Let $A \in B(T_l)$, $B \in B(T_r)$ as in (3), $\alpha, \beta \in A(\Omega)\backslash\{\mathrm{id}\}$, and $F \in F(\Omega)$ with $S(\alpha) \cap F \subseteq S(\beta)$. Put $C = (F \cap T_l^-)_l \cap A$, $D = (F \cap T_r^-)_r \cap B$. Then $C \in B(T_l)$, $D \in B(T_r)$ by Lemma 3.9. Furthermore, we have $(F(\alpha))_l \cap T_l^- \cap$

$C \subseteq (F(\beta))_l$ and $(F(\alpha))_l \cap \mathrm{Mid}(T_l) \cap C = \emptyset$, hence $(F(\alpha))_l \cap C \subseteq (F(\beta))_l$ and, symmetrically, $(F(\alpha))_r \cap D \subseteq (F(\beta))_r$. Now $\alpha \in (\beta)$ by Theorem 3.6.

Now we want to use Corollary 3.16 to show that in general in Theorem 3.6 condition (2ii) cannot be left out. For instance, it is easy to construct a tree $(T_R, \leqq_R) \in \mathscr{T}_R$ such that all maximal linearly ordered subsets of $(T_R, \leqq_R)$ are infinite and $|M_P| = 3$ whenever $P \subseteq T_R$ is a countable chain not containing a maximal element. Then by Theorem 2.11 there exists a doubly homogeneous chain $(\Omega, \leqq)$ such that $(T_r(\Omega), \leqq_r) = (T_R, \leqq_R)$. Hence condition (2), and thus also (1), of Corollary 3.16 is violated. We give a further example:

EXAMPLE 3.17. Let $\alpha \geqq 1$ be an ordinal and assume that $(\Omega, \leqq)$ has cardinality $\aleph_\alpha$ and is an η_α-set, i.e. a chain such that whenever $A, B \subseteq \Omega$ with $A < B$ and $|A|, |B| < \aleph_\alpha$, there exists $z \in \Omega$ with $A < z < B$. In particular, Ω is doubly homogeneous (Chang and Keisler [4; Prop. 5.1.14]). Let $T = T_r(\Omega)$. Since $\mathrm{cof}(\Omega) \neq \aleph_0$ and any $a \in \bar{\Omega}$ with $\mathrm{cof}(a) = \aleph_0$ ($\mathrm{coi}(a) = \aleph_0$) satisfies $\mathrm{coi}(a) \neq \aleph_0$ ($\mathrm{cof}(a) \neq \aleph_0$), we obtain $\omega \in I_r$ and $T_\omega \cap \mathrm{Mid}(T) \cap B \neq \emptyset$ for any $B \in B(T)$. Hence condition (2) of Corollary 3.16 does not hold.

§4. Constructing doubly homogeneous chains

All of this section is devoted to proving Theorem 2.11. In our proof, we will construct the chain $(\Omega, \leqq)$ as the union of a "good λ-system" of "good λ-sets" $(\Omega_i, \leqq)$. We will first define these notions and establish some properties of such sets.

DEFINITION 4.1. Let λ be a cardinal. A chain $(M, \leqq)$ is called a good λ-set, if the following conditions are satisfied:

(1) $|M| = \lambda$, and $(M, \leqq)$ is dense and unbounded;

(2) $\mathrm{cot}(a) = \aleph_0$ for each $a \in M$;

(3) for all elements $x, y \in M$ with $x < y$ there exists a set $A \subseteq [x, y]_{\bar{M}} \backslash M$ such that $|A| = \lambda$ and $\mathrm{cot}(a) = \aleph_0$ for each $a \in A$.

Obviously, any good $\aleph_0$-set is isomorphic to $\mathbf{Q}$, the set of all rationals. Now we deal with the existence problem of good λ-sets for arbitrary cardinals λ:

LEMMA 4.2. *Let λ be a cardinal. Then there exists a good λ-set $(M, \leqq)$ of countable coterminality.*

PROOF. Let K be the set of all sequences $(a_i)_{i \in \omega}$ of ω ordinals a_i satisfying $1 \leqq a_i < \lambda$ which are eventually constant and even, i.e. there exists an $i \in \omega$ such that for each $j \geqq i$, $a_j = a_i$ is an even ordinal. We define a linear order on K by ordering K lexicographically. Let $L \subseteq K$ consist of all those sequences which are eventually equal to 2. First we show:

(+) For all $a \in K$ there are $y_k, z_k \in L$ such that $y_k < a < z_k$ for each $k \in \mathbf{N}$ and $a = \sup\{y_k \mid k \in \mathbf{N}\} = \inf\{z_k \mid k \in \mathbf{N}\}$ in K.

Indeed, let $a = (a_i)_i \in K$. Fix $j \in \omega$ such that for each $n \geqq j$, $a_n = a_j$ is an even ordinal. For each $k \in \mathbf{N}$, define $y_k = (y_{ki})_i$, $z_k = (z_{ki})_i \in L$ such that $y_{ki} = z_{ki} = a_i$ if $i < j + k$, $y_{k,j+k} = 1$, $z_{k,j+k} = a_{j+k} + 1$, $y_{ki} = z_{ki} = 2$ if $i > j + k$. This establishes (+). Next we show:

(+ +) Whenever $x, y \in L$ satisfy $x < y$, there exists a set $A \subseteq [x, y]_K \backslash L$ of cardinality λ.

Indeed, let $x = (x_i)_i$, $y = (y_i)_i \in L$ and let $j = \min\{i \in \omega \mid x_i < y_i\}$. For each ordinal ν with $1 \leqq \nu < \lambda$, define an element $a_\nu = (a_{\nu i})_i \in K$ by putting $a_{\nu i} = x_i$ if $i \leqq j + 1$, $a_{\nu, j+2} = x_{j+2} + \nu$, and $a_{\nu,i} = 4$ whenever $i \geqq j + 3$. Then the set $A = \{a_\nu \mid 1 \leqq \nu < \lambda\}$ satisfies the assertion of (+ +).

Now define $a = (a_i)_i$, $b = (b_i)_i \in K$ by $a_i = 2$, $b_i = 4$ for each $i \in \omega$, and put $M = (a, b)_L$. Clearly $|M| = \lambda$, and (+) and (+ +) immediately show that $(M, \leqq)$ is a good λ-set of countable coterminality.

NOTATION 4.3. Let $(\Omega_i, \leqq)$ $(i = 1, 2)$ be dense unbounded chains such that $(\Omega_1, \leqq) \subseteq (\Omega_2, \leqq)$ and Ω_1 is unbounded in Ω_2. Suppose $a \in \Omega_2$, $A = \{x \in \Omega_1 \mid x < a\}$, and $B = \{y \in \Omega_1 \mid a < y\}$. If we have $a = \sup A = \inf B$ in Ω_2, and hence in particular A (B) is unbounded above (below), then we usually identify a with $\sup_{\bar{\Omega}_1} A = \inf_{\bar{\Omega}_1} B$ as elements of $\bar{\Omega}_1$; thus $a \in \bar{\Omega}_1$. Now let $a \in \bar{\Omega}_1$ or $a \in \bar{\Omega}_2$, and $\Omega_2 \subseteq Z \subseteq \bar{\Omega}_2$. Then we put

$$\operatorname{Ded}(a, \Omega_1, Z) = \{z \in Z \mid \{x \in \Omega_1 \mid x < a\} < z < \{y \in \Omega_1 \mid a < y\}\}.$$

A pair (A, B) of non-empty subsets $A, B \subseteq \Omega_1$ is called a *jump of* Ω_1 iff $\Omega_1 = A \dot{\cup} B$, $A < B$, and A (B) is unbounded above (below), i.e. iff $A = \{z \in \Omega_1 \mid z < a\}$, $B = \{z \in \Omega_1 \mid a < z\}$ for some $a \in \bar{\Omega}_1 \backslash \Omega_1$. A jump (A, B) of Ω_1 is called *empty in* Ω_2 iff there exists no $x \in \Omega_2$ with $A < x < B$, i.e. iff $\operatorname{Ded}(\sup A, \Omega_1, \Omega_2) = \varnothing$, and *non-empty*, if it is not empty. Let $\operatorname{Jump}(\Omega_1, \Omega_2)$ be the set of all jumps of Ω_1 which are non-empty in Ω_2; there exists a (natural) bijection from $\operatorname{Jump}(\Omega_1, \Omega_2)$ onto $\{\operatorname{Ded}(a, \Omega_1, \Omega_2) \mid a \in \Omega_2\}$. Note that we have $\operatorname{Ded}(a, \Omega_1, \Omega_2) = \{a\}$ for each $a \in \Omega_1$ iff for each $z \in \Omega_2 \backslash \Omega_1$ the pair $(\{x \in \Omega_1 \mid x < z\}, \{y \in \Omega_1 \mid z < y\})$ is a jump of Ω_1. As a further example to this notation we remark that we have $\Omega_2 \subseteq \bar{\Omega}_1$ iff $\mathrm{Ded}(a, \Omega_1, \Omega_2) = \{a\}$ for each $a \in \Omega_2$.

DEFINITION 4.4. Let $(\Omega_1, \leqq)$, $(\Omega_2, \leqq)$ be dense unbounded chains. Then Ω_1 is a good subset of Ω_2, i.e. $\Omega_1 \subseteq \Omega_2$ (good), if $(\Omega_1, \leqq) \subseteq (\Omega_2, \leqq)$, Ω_1 is unbounded in Ω_2, and $\mathrm{Ded}(a, \Omega_1, \Omega_2) = \{a\}$ for each $a \in \Omega_1$.

Clearly, this relation is reflexive and antisymmetrical on the class of dense unbounded chains. In the next two lemmas we show that it is also transitive and has a closure property for unions of dense unbounded chains which are good subsets of each other.

LEMMA 4.5. *Let* $(\Omega_i, \leqq)$, $i = 1, 2, 3$, *be dense unbounded chains such that* $\Omega_1 \subseteq \Omega_2$ *(good) and* $\Omega_2 \subseteq \Omega_3$ *(good). Then* $\Omega_1 \subseteq \Omega_3$ *(good)*

PROOF. Clearly, $(\Omega_1, \leqq) \subseteq (\Omega_3, \leqq)$ and Ω_1 is unbounded in Ω_3. Let $a \in \Omega_1$. We claim that $\mathrm{Ded}(a, \Omega_1, \Omega_3) = \{a\}$. Let $A = \{x \in \Omega_1 \mid x < a\}$ and $B = \{y \in \Omega_1 \mid a < y\}$. By $\mathrm{Ded}(a, \Omega_1, \Omega_2) = \{a\}$ we obtain that no $z \in \Omega_2$ satisfies $A < z < a$ or $a < z < B$. Hence $\mathrm{Ded}(a, \Omega_1, \Omega_3) = \mathrm{Ded}(a, \Omega_2, \Omega_3) = \{a\}$.

NOTATION. If I is a set of ordinals and $(A_i, \leqq_i)$ $(i \in I)$ are chains such that $(A_i, \leqq_i) \subseteq (A_j, \leqq_j)$ whenever $i < j$, $i, j \in I$, we put $(A, \leqq) = \bigcup_{i \in I} (A_i, \leqq_i)$ iff $A = \bigcup_{i \in I} A_i$ and for all $i \in I$, $a, b \in A_i$ we have $a \leqq b$ iff $a \leqq_i b$.

LEMMA 4.6. *Let j be a limit-ordinal,* $(\Omega_i, \leqq)$ $(i < j)$ *dense unbounded chains such that* $(\Omega_i, \leqq) \subseteq (\Omega_k, \leqq)$ *whenever* $i < k < j$, *and* $(\Omega, \leqq) = \bigcup_{i<j} (\Omega_i, \leqq)$. *Suppose that for some* $i < j$ *and* $a \in \bar{\Omega}_i$ *we have* $\mathrm{Ded}(a, \Omega_i, \Omega_k) \subseteq \{a\}$ *whenever* $i < k < j$. *Then*:

(a) $\mathrm{Ded}(a, \Omega_i, \Omega) \subseteq \{a\}$;

(b) *if* $A, B \subseteq \Omega_i$ *satisfy* $a = \sup A = \inf B$ *in* $\bar{\Omega}_i$, *then also* $a = \sup A = \inf B$ *in* $\bar{\Omega}$, *and* $a \in \bar{\Omega}$;

(c) $\mathrm{cof}_{\bar{\Omega}}(a) = \mathrm{cof}_{\bar{\Omega}_i}(a)$, $\mathrm{coi}_{\bar{\Omega}}(a) = \mathrm{coi}_{\bar{\Omega}_i}(a)$.

In particular, if $\Omega_i \subseteq \Omega_k$ *(good) whenever* $i < k < j$, *then* $\Omega_i \subseteq \Omega$ *(good) for any* $i < j$.

PROOF. Here (a) is clear by $\mathrm{Ded}(a, \Omega_i, \Omega) = \bigcup_{i<k<j} \mathrm{Ded}(a, \Omega_i, \Omega_k) \subseteq \{a\}$, (b) by (a), since no $x \in \Omega$ satisfies $A < x < a$ or $a < x < B$, (c) immediately by (b), and the final statement by (a).

Next we wish to find conditions for a chain of good λ-sets $(\Omega_i, \leqq)$, $i < j$, which are sufficient to imply that $(\Omega, \leqq) = \bigcup_{i<j} (\Omega_i, \leqq)$ is again a good λ-set.

DEFINITION 4.7. Let λ be a cardinal, $(\Omega_i, \leqq)$ good λ-sets, and $T_i \subseteq \bar{\Omega}_i \backslash \Omega_i$, for $i = 1, 2$. Then we call (Ω_1, T_1) a good subsystem of (Ω_2, T_2), i.e. $(\Omega_1, T_1) \subseteq (\Omega_2, T_2)$ (good), if the following conditions are satisfied:

(I) $\Omega_1 \subseteq \Omega_2$ (good), $T_1 \subseteq T_2$, and $(\Omega_1 \cup T_1, \leqq) \subseteq (\Omega_2 \cup T_2, \leqq)$.

(II) $|\mathrm{Jump}(\Omega_1, \Omega_2)| < \lambda$.

(III) If $a \in T_1$, then $\mathrm{Ded}(a, \Omega_1, \Omega_2 \cup T_2) = \{a\}$.

(IV) If $a \in T_2 \backslash T_1$, then $\mathrm{Ded}(a, \Omega_1, \Omega_2) \neq \varnothing$.

(V) If $a \in \Omega_2 \backslash \Omega_1$, $A = \{z \in \Omega_1 \mid z < a\}$, $B = \mathrm{Ded}(a, \Omega_1, \Omega_2)$, and $C = \{z \in \Omega_1 \mid a < z\}$, then

(i) B is unbounded,

(ii) $\mathrm{cof}(A) = \mathrm{cot}(B) = \mathrm{coi}(C) = \aleph_0$,

(iii) there are $x, y \in T_2$ such that, in $\bar{\Omega}_2$, we have $A < x < B < y < C$.

(VI) Whenever $i \in \{1, 2\}$ and $x, y \in \Omega_i$ with $x < y$, there exists a set $A \subseteq [x, y]_{\bar{\Omega}_i} \backslash (\Omega_i \cup T_i)$ such that $|A| = \lambda$ and $\mathrm{cot}(a) = \aleph_0$ for each $a \in A$.

Here, (III) says that if $a \in T_1$, then no $b \in \Omega_2 \cup T_2$ with $b \neq a$ realizes the same Dedekind cut in Ω_1 as does a; in this respect a can be thought of as a "forbidden point" of $\bar{\Omega}_1 \backslash \Omega_1$. Condition (V) says that whenever (A, C) is a jump of Ω_1 and we obtain Ω_2 by "inserting" (precisely defined later) a set B into Ω_1 between A and C, thus $A < B < C$ in Ω_2, then B is unbounded with countable coterminality, and the points of $\bar{\Omega}_2 \backslash \Omega_2$ corresponding to the jumps $(A, B \cup C)$, $(A \cup B, C)$ of Ω_2 become forbidden points of $\bar{\Omega}_2 \backslash \Omega_2$, i.e. elements of T_2. Condition (IV) is sort of converse to (V). Note that by (IV) we have $\mathrm{Jump}(\Omega_1, \Omega_2) = \mathrm{Jump}(\Omega_1, \Omega_2 \cup (T_2 \backslash T_1))$. Condition (VI) sharpens condition (3) of Definition 4.1.

REMARK. In the situation of (4.7) and the notation of condition (V), we always have $x = \sup A = \inf B$ and $y = \inf C = \sup B$ in $\bar{\Omega}_2$.

PROOF. We only prove the first row of equations. First note that $x \in \underrightarrow{\lim}(\Omega_2) \cap \underleftarrow{\lim}(\Omega_2)$ by $x \in T_2 \subseteq \bar{\Omega}_2$. But any $w \in \Omega_2$ with $A < w$ satisfies either $w \in B$ or $B < w$, hence $x < w$ by condition (V). Thus $x \in \underrightarrow{\lim}(\Omega_2)$ implies $x = \sup A$. Similarly, $x = \inf B$ follows from $x \in \underleftarrow{\lim}(\Omega_2)$.

The following lemmas (4.8) and (4.10) establish an analogue to (4.5) and (4.6) for the relation defined in (4.7).

LEMMA 4.8. *Let λ be a cardinal, $(\Omega_i, \leqq)$ good λ-sets and $T_i \subseteq \bar{\Omega}_i \backslash \Omega_i$ $(i = 1, 2, 3)$. If $(\Omega_1, T_1) \subseteq (\Omega_2, T_2)$ (good) and $(\Omega_2, T_2) \subseteq (\Omega_3, T_3)$ (good), then $(\Omega_1, T_1) \subseteq (\Omega_3, T_3)$ (good).*

PROOF. We have $\Omega_1 \subseteq \Omega_3$ (good) by Lemma 4.5, so conditions (I), (VI) of (4.7) are satisfied. For condition (II), let (A, B) be any non-empty jump of Ω_1 in Ω_3. If there exists $z \in \Omega_2$ with $A < z < B$, then (A, B) is a non-empty jump of Ω_1 in Ω_2. Otherwise, if no $z \in \Omega_2$ satisfies $A < z < B$, let

$$A^* = \{x \in \Omega_2 \mid \exists a \in \Omega_1 : x \leqq a\} \quad \text{and} \quad B^* = \{x \in \Omega_2 \mid \exists b \in \Omega_1 : b \leqq x\};$$

then $A \subseteq A^*$, $B \subseteq B^*$, and $\sup A = \sup A^*$, $\inf B = \inf B^*$ in $\bar{\Omega}_3$, and (A^*, B^*) is a non-empty jump of Ω_2 in Ω_3. This shows

$$\operatorname{Jump}(\Omega_1, \Omega_3)| \leqq |\operatorname{Jump}(\Omega_1, \Omega_2)| + |\operatorname{Jump}(\Omega_2, \Omega_3)| < \lambda.$$

For (III), let $a \in T_1$. Then $\operatorname{Ded}(a, \Omega_1, \Omega_2 \cup T_2) = \{a\}$ implies $\operatorname{Ded}(a, \Omega_1, \Omega_3 \cup T_3) = \operatorname{Ded}(a, \Omega_2, \Omega_3 \cup T_3) = \{a\}$. For (IV), let $a \in T_3 \backslash T_1$ and $A = \operatorname{Ded}(a, \Omega_1, \Omega_3)$. If $a \in T_2$, $(\Omega_1, T_1) \subseteq (\Omega_2, T_2)$ (good) implies $\varnothing \neq \operatorname{Ded}(a, \Omega_1, \Omega_2) \subseteq A$. If $a \notin T_2$, $(\Omega_2, T_2) \subseteq (\Omega_3, T_3)$ (good) yields $\varnothing \neq \operatorname{Ded}(a, \Omega_2, \Omega_3) \subseteq A$. Hence $A \neq \varnothing$ in any case.

For (V), let $a \in \Omega_3 \backslash \Omega_1$, $A = \{z \in \Omega_1 \mid z < a\}$, and $B = \operatorname{Ded}(a, \Omega_1, \Omega_3)$. We first show $\operatorname{cof}(A) = \aleph_0$. If $a \in \Omega_2$, this is clear by $(\Omega_1, T_1) \subseteq (\Omega_2, T_2)$ (good). So suppose $a \notin \Omega_2$. Let $D = \{z \in \Omega_2 \mid z < a\}$. First assume that for each $z \in D$ there exists a $w \in A$ with $z \leqq w$. Then clearly $\operatorname{cof}(A) = \operatorname{cof}(D) = \aleph_0$ by $(\Omega_2, T_2) \subseteq (\Omega_3, T_3)$ (good). On the other hand, if there exists an element $d \in D$ with $A < d$, we obtain $A = \{z \in \Omega_1 \mid z < d\}$ and so $\operatorname{cof}(A) = \aleph_0$ by $(\Omega_1, T_1) \subseteq (\Omega_2, T_2)$ (good).

Next we show the part of (V) concerning B. Let $b = \inf B \in \bar{\Omega}_3$ and $E = B \cap \Omega_2$. Note that if $E \neq \varnothing$, then $E = \operatorname{Ded}(e, \Omega_1, \Omega_2)$ for any $e \in E$, and hence $\inf E \notin E$ and $\operatorname{coi}(E) = \aleph_0$ by $(\Omega_1, T_1) \subseteq (\Omega_2, T_2)$ (good). We first show that B is unbounded below. By way of contradiction, assume $b \in B$. If $b \in \Omega_2$, we obtain $\inf E = b \in \Omega_2$, a contradiction. If $b \notin \Omega_2$, then the set $\operatorname{Ded}(b, \Omega_2, \Omega_3)$ is unbounded and hence contains elements smaller than b, contradicting $b = \inf B$. This shows $b \notin B$ as claimed.

Now if $E \neq \varnothing$ and $\inf E = b$ in $\bar{\Omega}_3$, then $\operatorname{coi}(B) = \operatorname{coi}(E) = \aleph_0$. Let $e \in E$. There exists $x \in T_2 \subseteq T_3$ with $\{z \in \Omega_1 \mid z < e\} < x < E$. Thus $x = \inf E = b$ and $A < x < B$ as claimed. On the other hand, if $E = \varnothing$ or $E \neq \varnothing$ and $b < \inf E$, there exists an element $f \in \Omega_3$ with $f < a$ and $(b, f]_{\Omega_3} \cap \Omega_2 = \varnothing$. Hence $(b, f]_{\Omega_3} \subseteq F = \operatorname{Ded}(f, \Omega_2, \Omega_3)$, $b = \inf F \in T_3$, $\operatorname{coi}(B) = \operatorname{coi}(F) = \aleph_0$, and $\{z \in \Omega_2 \mid z < f\} < \inf F < F$ by $(\Omega_2, T_2) \subseteq (\Omega_3, T_3)$ (good). But any $z \in \Omega_1$ with $z < a$ satisfies $z < f$, thus $A < b < B$, establishing the first half of our claim for B. The rest of (V) follows by a symmetry-argument.

DEFINITION 4.9. Let λ be a cardinal, $j<\lambda$ an ordinal, $(\Omega_i, \leqq)$ good λ-sets and $T_i \subseteq \bar{\Omega}_i \backslash \Omega_i$ for each $i<j$. If $(\Omega_i, T_i) \subseteq (\Omega_k, T_k)$ (good) whenever $i<k<j$, and $\Omega_k = \bigcup_{i<k} \Omega_i$, $T_k = \bigcup_{i<k} T_i$, $(\Omega_k \cup T_k, \leqq) = \bigcup_{i<k}(\Omega_i \cup T_i, \leqq)$ whenever $k<j$ is a limit-ordinal, then $(\Omega_i, T_i)_{i<j}$ is called a *good λ-system.*

LEMMA 4.10. *Let λ be an uncountable regular cardinal, $j<\lambda$ a limit-ordinal, $(\Omega_i, \leqq)$ good λ-sets and $T_i \subseteq \bar{\Omega}_i \backslash \Omega_i$ for each $i<j$, such that $(\Omega_i, T_i)_{i<j}$ is a good λ-system. If $\Omega_j = \bigcup_{i<j} \Omega_i$, $T_j = \bigcup_{i<j} T_i$, and $(\Omega_j \cup T_j, \leqq) = \bigcup_{i<j}(\Omega_i \cup T_i, \leqq)$, then $(\Omega_j, \leqq)$ is a good λ-set, $T_j \subseteq \bar{\Omega}_j \backslash \Omega_j$, and $(\Omega_i, T_i) \subseteq (\Omega_j, T_j)$ (good) for each $i<j$. Hence $(\Omega_i, T_i)_{i<j+1}$ is a good λ-system.*

PROOF. First we show that $(\Omega_j, \leqq)$ is a good λ-set. Here, condition (1) of (4.1) is obvious. For (4.1)(2), let $a \in \Omega_j$. Choose $i<j$ with $a \in \Omega_i$. Now Lemma 4.6 shows $\operatorname{cot}_{\Omega_j}(a) = \operatorname{cot}_{\Omega_i}(a) = \aleph_0$. Let us now check conditions (II), (VI) and hence also (4.1)(3). Let $i<j$. Since

$$\operatorname{Jump}(\Omega_i, \Omega_j \cup (T_j \backslash T_i)) = \bigcup_{i<k<j} \operatorname{Jump}(\Omega_i, \Omega_k \cup (T_k \backslash T_i)) = \bigcup_{i<k<j} \operatorname{Jump}(\Omega_i, \Omega_k),$$

we have $|\operatorname{Jump}(\Omega_i, \Omega_j)| \leqq \leqq |\operatorname{Jump}(\Omega_i, \Omega_j \cup (T_j \backslash T_i)| < \lambda$, showing (II). Now let $x, y \in \Omega_i$ with $x<y$. Choose $A \subseteq [x, y]_{\bar{\Omega}_i} \backslash (\Omega_i \cup T_i)$ such that $|A| = \lambda$ and $\operatorname{cof}_{\Omega_i}(a) = \aleph_0$ for each $a \in A$. Since $|\operatorname{Jump}(\Omega_i, \Omega_j \cup (T_j \backslash T_i))| < \lambda$, there exists a set $B \subseteq A$ with $|B| = \lambda$ such that for each $a \in B$, the pair (A_a, B_a), where $A_a = \{z \in \Omega_i \mid z<a\}$, $B_a = \{z \in \Omega_i \mid a<z\}$, is an empty jump of Ω_i in $\Omega_j \cup T_j$, thus $a = \sup A_a = \inf B_a$ in $\bar{\Omega}_j$ and $a \in \bar{\Omega}_j \backslash (\Omega_j \cup T_j)$. This shows $B \subseteq \bar{\Omega}_j \backslash (\Omega_j \cup T_j)$, and by Lemma 4.6 we obtain $\operatorname{cot}_{\Omega_j}(a) = \operatorname{cot}_{\Omega_i}(a) = \aleph_0$ for each $a \in B$.

Now we claim $T_j \subseteq \bar{\Omega}_j \backslash \Omega_j$. Indeed, whenever $a \in T_i \subseteq \bar{\Omega}_i \backslash \Omega_i$ $(i<j)$, we have $a \in \bar{\Omega}_j \backslash \Omega_j$ by Lemma 4.6, and

$$\operatorname{Ded}(a, \Omega_i, \Omega_j \cup T_j) = \bigcup_{i<k<j} \operatorname{Ded}(a, \Omega_i, \Omega_k \cup T_k) = \{a\},$$

thus establishing also (4.7)(III). Lemma 4.6 also implies $\Omega_i \subseteq \Omega_j$ (good) for each $i<j$. To finish the proof that $(\Omega_i, T_i) \subseteq (\Omega_j, T_j)$ (good) for each $i<j$, it remains to check (4.7)(IV) and (V). Let $i<j$. For (IV), let $a \in T_j \backslash T_i$ and $k<j$ such that $a \in T_k$. Then

$$\operatorname{Ded}(a, \Omega_i, \Omega_j) \supseteq \operatorname{Ded}(a, \Omega_i, \Omega_k) \neq \varnothing \quad \text{by} \quad (\Omega_i, T_i) \subseteq (\Omega_k, T_k) \text{ (good).}$$

Finally, we prove (V). Let $a \in \Omega_j \backslash \Omega_i$ and choose $k<j$ such that $a \in \Omega_k$, thus $i<k<j$. Put $A = \{z \in \Omega_i \mid z<a\}$, $B = \operatorname{Ded}(a, \Omega_i, \Omega_k)$, $C = \operatorname{Ded}(a, \Omega_i, \Omega_j)$.

Since $(\Omega_i, T_i) \subseteq (\Omega_k, T_k)$ (good), we obtain $\mathrm{cof}(A) = \aleph_0$, and there is an element $x \in T_k \subseteq T_j$ such that, in $\bar{\Omega}_k$, we have $A < x < B$, B is unbounded below and $\mathrm{coi}(B) = \aleph_0$. Hence $x = \sup A = \inf B$ in $\bar{\Omega}_k$ and thus also in $\bar{\Omega}_j$ by Lemma 4.6. Since $A < C$ and $x \notin \Omega_j$, we obtain $A < x < C$ and thus $x = \inf B = \inf C$ by $B \subseteq C$. Hence C is unbounded below and $\mathrm{coi}(C) = \mathrm{coi}(B) = \aleph_0$. Now a symmetry-argument implies the rest of (4.7)(V).

Another essential technique for the proof of Theorem 2.11 is the following natural *inserting process*. Suppose that $(M, \leqq)$ is a chain, $(M_i, \leqq_i)$ are pairwise disjoint chains such that $M \cap M_i = \emptyset$ for each $i \in I$, and $M' = M \dot{\cup} \bigcup_{i \in I} M_i$. Under certain additional assumptions we want to define a linear order $\leqq'$ on M' which extends $\leqq$ and $\leqq_i$ for each $i \in I$. First let $a_i, b_i \in M$ $(\emptyset \neq A_i, B_i \subseteq M)$ for each $i \in I$ such that $a_i < b_i$ $(A_i < B_i)$ and no $x \in M$ satisfies $a_i < x < b_i$ $(A_i < x < B_i)$. Whenever $i, j \in I$, $i \neq j$, assume that either $b_i \leqq a_j$ or $b_j \leqq a_i$ (there exist either $a \in A_j$, $b \in B_i$ such that $b \leqq a$, or $a \in A_i$, $b \in B_j$ such that $b \leqq a$). Then we say that we *insert M_i into M between a_i and b_i (A_i and B_i) for each $i \in I$* if we define the order $\leqq'$ on M' in the natural way such that

$$(M, \leqq) \subseteq (M', \leqq'), \quad (M_i, \leqq_i) \subseteq (M', \leqq'), \quad \text{and} \quad a_i <' M_i <' b_i \quad (A_i <' M_i <' B_i)$$

for each $i \in I$. Now assume in addition that $(M, \leqq)$ is dense and $\{a_i \mid i \in I\} \subseteq \bar{M} \backslash M$. Then we *insert M_i into M at a_i for each $i \in I$* if we insert M_i into M between $\{x \in M \mid x < a_i\}$ and $\{y \in M \mid a_i < y\}$ for each $i \in I$.

As an example we remark that if $(\Omega, \leqq)$ is a dense unbounded chain, we obtain $(\bar{\Omega}, \leqq)$ by inserting, for each jump (A, B) of Ω, an element $a = a(A, B)$ into Ω between A and B.

We will apply such an inserting-argument for the following

Remark 4.11. For each tree $(T, \leqq_T) \in \mathcal{T}$ there exists a linear order $\leqq$ on $T^{\sim} = T \backslash \mathrm{Mid}(T)$ — as described below — which is called the *associated order on $T^{\sim}$*.

Definition of $\leqq$. For each ordinal i, let

$$T_i^{\sim} = \{x \in T^{\sim} \mid (\{y \in T \mid y <_T x\}, \leqq_T) \cong i\} \quad \text{and} \quad S_i^{\sim} = T_i^{\sim} \backslash \bigcup_{j < i} T_j^{\sim},$$

and let $h = h(T)$ be the least ordinal α such that $T_\alpha^{\sim} = \emptyset$. Thus we have

$$T_0^{\sim} = \min(T, \leqq_T), \qquad T_{i+1}^{\sim} = T_i^{\sim} \dot{\cup} \bigcup_{a \in S_i^{\sim}} M_a$$

whenever $i < h$ (where M_a is, as in §2, the set of all minimal elements of $T^{\sim}$ above a), and

$$T_i^{\sim} = \bigcup_{j<i} T_j^{\sim} \,\dot{\cup}\, \dot{\bigcup} \{M_P \cap T^{\sim} \mid P \text{ maximal path in } \bigcup_{j<i} T_j^{\sim}\}$$

whenever $i < h$ is a limit-ordinal, and h is the height of T. By induction, we now define linear orders $\leqq$ on $T_i^{\sim}$ for each $i < h$ such that $(T_i^{\sim}, \leqq) \subseteq (T_j^{\sim}, \leqq)$ whenever $i < j < h$. Then we put $(T^{\sim}, \leqq) = \bigcup_{i<h}(T_i^{\sim}, \leqq)$. Of course, $T_0^{\sim}$ is trivially ordered.

So suppose $1 \leqq i < h$ and let $(T_j^{\sim}, \leqq)$ already have been defined for each $j < i$. First assume that $i = k+1$ for some ordinal k. Let $Z_k = \{a \in S_k^{\sim} \mid M_a \neq \emptyset\}$. For each $a \in Z_k$ we define a linear order on $M_a^+ = M_a \dot{\cup} \{a\}$ by putting $(M_a, \leqq_a) \subseteq (M_a^+, \leqq)$ and $M_a < a$ $(a < M_a)$ if $a^\Phi = 3$ $(a^\Phi = 1)$, i.e. if M_a is well-ordered (inversely well-ordered). Now we define $(T_i^{\sim}, \leqq)$ such that $(T_k^{\sim}, \leqq) \subseteq (T_i^{\sim}, \leqq)$ and, for each $a \in Z_k$ and $z \in T_i^{\sim} \setminus M_a^+$, we have $(M_a^+, \leqq) \subseteq (T_i^{\sim}, \leqq)$ and $z < M_a^+$ $(M_a^+ < z)$ in $(T_i^{\sim}, \leqq)$ whenever

either $z \in T_k^{\sim}$ and $z < a$ $(a < z)$ in $(T_k^{\sim}, \leqq)$,

or $z \in M_b^+$ for some $b \in Z_k$ with $b < a$ $(a < b)$ in $(T_k^{\sim}, \leqq)$;

thus the points $a \in Z_k$ are simply "replaced" by the chains $(M_a^+, \leqq)$.

It remains to consider the case that i is a limit-ordinal. Let P be a maximal path in $\bigcup_{j<i} T_j^{\sim}$ and $P_n = \{a \in P \mid a^\Phi = n\}$ $(n = 1, 3)$; we may assume $P_1 < P_3$ as induction hypothesis. For $M_P^{\sim} = M_P \cap T^{\sim} \subseteq S_i^{\sim}$, we have either $M_P^{\sim} = \{a_P\}$ or $M_P^{\sim} = \{a_P, c_P\}$ with $a_P^\Phi = 1$, $c_P^\Phi = 3$, and we define a linear order on $M_P^{\sim}$, correspondingly, either trivially or by putting $a_P < c_P$. Then define $(T_i^{\sim}, \leqq)$ by inserting, for each such path P, $M_P^{\sim}$ into $\bigcup_{j<i}(T_j^{\sim}, \leqq)$ between P_1 and P_3, i.e. we put $P_1 < M_P^{\sim} < P_3$. This defines our linear order $\leqq$ on $T^{\sim}$.

The subsequent remarks to the construction above will be used at the end (Part IV) of the proof of Theorem 2.11.

REMARK 4.12. (a) We have $\min(T, \leqq_T) = \max(T^{\sim}, \leqq)$ $(= \min(T^{\sim}, \leqq))$ iff $T \in \mathcal{T}_R$ $(T \in \mathcal{T}_L)$.

(b) Whenever $i + 3 < h$, $a_k \in T_{i+k}^{\sim}$ $(k = 0, 1, 2, 3)$, $a_0^\Phi = 1$, and $a_k <_T a_{k+1}$ $(k = 0, 1, 2)$, then we have $a_0 < a_2 < a_3 < a_1$ in $(T^{\sim}, \leqq)$.

(c) If $a \in T^{\sim}$, $a^\Phi = 3$, and $M_a \neq \emptyset$, then in $(T^{\sim}, \leqq)$ we have $M_a < a$, $(M_a, \leqq)$ is an uncountable well-ordered set isomorphic to $|M_a|$, and $M_a \dot{\cup} \{a\}$ is closed in $(T^{\sim}, \leqq)$ such that $a = \sup M_a$.

(d) If $a \in T^{\sim}$, $a^{\Phi}=3$, $M_a=\emptyset$, and $a \neq \min(T, \leqq_T)$, then there exists a maximal element b in $(T^{\sim}, \leqq)$ with $b<a$.

(e) Suppose that $i<h$ is a limit-ordinal, P a maximal path in $\bigcup_{j<i} T_j^{\sim}$ and $P_n=\{a \in P \mid a^{\Phi}=n\}$ $(n=1,3)$. If $M_P=\{a_P\}$, we have $P_1<a_P<P_3$ and $a_P=\sup P_1=\inf P_3$ in $(T^{\sim}, \leqq)$. If $M_P=\{a_P, b_P, c_P\}$ with $a_P^{\Phi}=1$, $b_P^{\Phi}=2$, $c_P^{\Phi}=3$, we have $P_1<a_P<c_P<P_3$ and $a_P=\sup P_1$, $c_P=\inf P_3$ in $(T^{\sim}, \leqq)$.

Proof. (a), (b), (c), (e) follow immediately from $T \in \mathcal{T}$ and the definition of $(T^{\sim}, \leqq)$. For (d), suppose $a \in T_i$ $(i<h)$. We distinguish between two cases.

If i is not a limit-ordinal, by $a \neq \min(T, \leqq_T)$ we have $a \in M_c$ for some $c \in T^{\sim}$. By $a^{\Phi}=3$ it follows that $c^{\Phi}=1$, hence $(M_c, \leqq)$ is inversely well-ordered, and there exists a maximal element b in $(M_c, \leqq)$ with $b<a$. But now any other element $z \in T^{\sim}$ with $z<a$ satisfies $z \leqq b$, since $M_a=\emptyset$.

On the other hand, if i is a limit-ordinal, let $P \subseteq \bigcup_{j<i} T_j^{\sim}$ be a maximal path with $P<a$. By $a \in M_P$ and since $a^{\Phi}=3$, we obtain $M_P=\{a_P, b_P, c_P\}$ with $a_P^{\Phi}=1$, $b_P^{\Phi}=2$, $c_P^{\Phi}=3$, and $a=c_P$. If $M_{a_P}=\emptyset$, put $b=a_P$, and if $M_{a_P} \neq \emptyset$, $(M_{a_P}, \leqq)$ is inversely well-ordered, hence let $b=\max(M_{a_P}, \leqq)$ to obtain the assertion.

After these preparations, we can now come to the

Proof of Theorem 2.11. Our proof can be divided into four main parts.

Part I. Preliminaries

If λ is countable, we obtain $|T_R|=|T_L|=1$. Hence we simply take $\Omega=\mathbf{Q}$ and identify $T_l(\Omega)=\{-\infty\}$ $(T_r(\Omega)=\{\infty\})$ with T_L (T_R), respectively. This shows that we may assume from now on that λ is uncountable and regular.

First, let $(T_R^{\sim}, \leqq)$ $((T_L^{\sim}, \leqq))$ be linearly ordered by the associated order on $T_R^{\sim}$ $(T_L^{\sim})$, respectively, according to Remark 4.11. Then we extend these orders in the natural way to obtain a chain $(T_L^{\sim} \dot{\cup} T_R^{\sim}, \leqq)$ satisfying $T_L^{\sim}<T_R^{\sim}$. Now we will constuct our chain $(\Omega, \leqq)$ such that, finally, $T_L^{\sim} \dot{\cup} T_R^{\sim} \subseteq \bar{\Omega} \backslash \Omega$.

We split $\lambda=\dot{\bigcup}_{k \in \lambda} \lambda_k$ with $|\lambda_k|=\lambda$ and $\min \lambda_k \geqq k$ for each $k \in \lambda$. By transfinite induction, we will define good λ-sets $(\Omega_i, \leqq)$ and sets $T_i \subseteq \bar{\Omega}_i \backslash \Omega_i$ for each $i \in \lambda$ such that $(\Omega_i, T_i)_{i \in \lambda}$ is a good λ-system and $T_0=(T_R^{\sim} \dot{\cup} T_L^{\sim}) \backslash \{\infty,-\infty\}$. (We will also have $|T_i| \leqq \lambda$ for each $i \in \lambda$, but this is for our purposes of no importance.) Furthermore, for each $k \in \lambda$ we will define a bijection $\phi_k: \lambda_k \rightarrow M_k$, where $M_k := \{(a, b, c, d) \in \Omega_k^4 \mid a<b<c<d\}$. Suppose $i \in \lambda_k \subseteq \lambda$, $i^{\phi_k}=(a, b, c, d) \in M_k$, say, and that Ω_{i+1}, T_{i+1} are already defined. Observe here that $k \leqq \min \lambda_k \leqq i$, thus $\Omega_k \subseteq \Omega_i \subseteq \Omega_{i+1}$. Then we will also define an isomorphism $\psi_i:[a, b]_{\Omega_{i+1}} \rightarrow[c, d]_{\Omega_{i+1}}$ with the following properties:

(+) If we extend ψ_i (in the uniquely determined way) to an isomorphism $\bar{\psi}_i : [a, b]_{\bar{\Omega}_{i+1}} \to [c, d]_{\bar{\Omega}_{i+1}}$, then

$$(T_{i+1} \cap [a, b]_{\bar{\Omega}_{i+1}})^{\bar{\psi}_i} = T_{i+1} \cap [c, d]_{\bar{\Omega}_{i+1}}.$$

(+ +) Whenever $j, m \in \lambda$, $j < i$, $j \in \lambda_m$, ψ_j has been defined as above, and $i^{\phi_k} = j^{\phi_m}$, then $\psi_i \mid [a, b]_{\Omega_{j+1}} = \psi_j$. (Here $\psi \mid A$ denotes the restriction of the mapping ψ to A.)

Then, as we will see later, $(\Omega, \leqq) = \bigcup_{i \in \lambda} (\Omega_i, \leqq)$ satisfies the assertion of the Theorem.

Part II. Construction of our good λ-system $(\Omega_i, T_i)_{i \in \lambda}$

Let $j \in \lambda$, and assume that $(\Omega_i, \leqq)$, T_i, ϕ_i for each $i < j$ and ψ_i for each $i \in \lambda$ with $i + 1 < j$, are already defined such that $(\Omega_i, T_i)_{i<j}$ is a good λ-system of the form described in Part I satisfying (+) and (+ +). We wish to define $(\Omega_j, \leqq)$, T_j, and, provided that $j = i + 1$ for some ordinal, also ψ_i, such that $(\Omega_i, T_i)_{i<j+1}$ is a good λ-system and (+) and (+ +) are satisfied correspondingly. Then we always choose ϕ_j to be any bijection from λ_j onto M_j.

Step 1. Assume $j = 0$

We choose a good λ-set C of countable coterminality by Lemma 4.2. For each pair (a, b) of points $a, b \in X = T_{\tilde{L}} \dot{\cup} T_{\tilde{R}}$ such that $a < b$ and $[a, b]_X = \{a, b\}$, choose a copy $C_{a,b}$ of C, let Ω_0 be the disjoint union of all these copies $C_{a,b}$ (note that by $m_L = \max(T_{\tilde{L}}, \leqq) < \min(T_{\tilde{R}}, \leqq) = m_R$ we obtain $\Omega_0 \supseteq C_{m_L, m_R}$), and define a linear order $\leqq$ on $\Omega_0 \dot{\cup} X$ by inserting $C_{a,b}$ into X between the points a and b. We put $T_0 = X \setminus \{\max(T_{\tilde{R}}, \leqq), \min(T_{\tilde{L}}, \leqq)\}$. ($T_0$ might be empty; then $\Omega_0 = C_{m_L, m_R}$. If $|T_{\tilde{R}}| = 1$, note for later purposes that $\mathrm{cof}(\Omega_0) = \mathrm{cof}(C_{m_L, m_R}) = \aleph_0$.) All we have to check is that $(\Omega_0, \leqq)$ is a good λ-set and satisfies condition (VI) of (4.7), but this is obvious.

Step 2. Assume that j is a limit-ordinal

We simply put $\Omega_j = \bigcup_{i<j} \Omega_j$, $T_j = \bigcup_{i<j} T_j$, and $(\Omega_j \cup T_j, \leqq) = \bigcup_{i<j} (\Omega_i \cup T_i, \leqq)$. Then Ω_j is a good λ-set, $T_j \subseteq \bar{\Omega}_j \setminus \Omega_j$, and $(\Omega_i, T_i)_{i<j+1}$ is a good λ-system by Lemma 4.10, finishing this case.

Step 3. Assume that $j = i + 1$ for some ordinal $i \in \lambda$

Let $k \in \lambda$ satisfy $i \in \lambda_k$; hence $k \leqq \min \lambda_k \leqq i$. Assume $i^{\phi_k} = X = (a, b, c, d) \in M_k$, say, thus $a, b, c, d \in \Omega_k \subseteq \Omega_i$ and $a < b < c < d$. Let Λ be the set of all $n \in \lambda$ with $n < i$ and $n \in \lambda_m$ for some $m \in \lambda$ such that $n^{\phi_m} = X$. We distinguish between three subcases, namely $\Lambda = \emptyset$, $\sup \Lambda \in \Lambda$, and $\sup \Lambda \notin \Lambda$.

Case 1. Assume $\Lambda = \varnothing$

The assumption guarantees that condition $(++)$ is empty. By condition (4.7)(VI), there are sets

$$A = \{a_n \mid n \in \mathbf{Z}\} \subseteq [a, b]_{\bar{\Omega}_i} \backslash (\Omega_i \cup T_i) \quad \text{and} \quad B = \{b_n \mid n \in \mathbf{Z}\} \subseteq [c, d]_{\bar{\Omega}_i} \backslash (\Omega_i \cup T_i)$$

such that $\mathrm{cot}(x) = \aleph_0$ for each $x \in A \cup B$, $a_n < a_{n+1}$ and $b_n < b_{n+1}$ for each $n \in \mathbf{Z}$, and $a = \inf A$, $b = \sup A$, $c = \inf B$, $d = \sup B$. For each $n \in \mathbf{Z}$, let C_n (C'_n) be a copy of $A_n = (a_n, a_{n+1})_{\Omega_i \cup T_i} \dot\cup \{a_n, a_{n+1}\}$ ($B_n = (b_n, b_{n+1})_{\Omega_i \cup T_i} \dot\cup \{b_n, b_{n+1}\}$) and π_n (π'_n) an isomorphism from A_n (B_n) onto C_n (C'_n). Put

$$\Omega_j \equiv \Omega_{i+1} = \Omega_i \dot\cup \dot{\bigcup_{n \in \mathbf{Z}}} (A_n \cap \Omega_i)^{\pi_n} \dot\cup \dot{\bigcup_{n \in \mathbf{Z}}} (B_n \cap \Omega_i)^{\pi'_n}$$

and

$$T_j = T_i \dot\cup \dot{\bigcup_{n \in \mathbf{Z}}} (A_n \backslash \Omega_i)^{\pi_n} \dot\cup \dot{\bigcup_{n \in \mathbf{Z}}} (B_n \backslash \Omega_i)^{\pi'_n}.$$

We define a linear order $\leqq$ on $\Omega_j \cup T_j$ by inserting, for each $n \in \mathbf{Z}$, C_n into $\Omega_i \cup T_i$ at b_n and C'_n into $\Omega_i \cup T_i$ at a_{n+1}. To be explicit, we thus have, in particular,

$$(b_{n-1}, b_n)_{\Omega_i \cup T_i} < C_n < (b_n, b_{n+1})_{\Omega_i \cup T_i}$$

and

$$a_n^{\pi_n} = \min C_n, \qquad a_{n+1}^{\pi_n} = \max C_n \in T_j$$

for each $n \in \mathbf{Z}$. It follows directly from this construction that Ω_j is a good λ-set and that $\Omega_i \subseteq \Omega_j$ (good). We claim $(\Omega_i, T_i) \subseteq (\Omega_j, T_j)$ (good). Condition (4.7)(I) is clear, and (II) is obvious since $|\mathrm{Jump}(\Omega_i, \Omega_j)| = |A \cup B| = \aleph_0$. Conditions (III), (IV), (VI) also follow directly from our construction. For (V), let $a \in \Omega_j \backslash \Omega_i$, thus, e.g., $a \in (A_n \cap \Omega_i)^{\pi_n} \subseteq C_n$ for some $n \in \mathbf{Z}$. Then (V)(i) follows from $\mathrm{Ded}(a, \Omega_i, \Omega_j) = (A_n \cap \Omega_i)^{\pi_n}$, and (V)(ii) from $\mathrm{cof}_{\Omega_{i+1}}(b_n) = \mathrm{cof}_{\Omega_i}(b_n) = \aleph_0 = \mathrm{coi}_{\Omega_{i+1}}(b_{n+1})$ and from $\mathrm{cot}_{\Omega_i}(A_n \cap \Omega_i) = \aleph_0$ since $\mathrm{coi}_{\Omega_i}(a_n) = \mathrm{cof}_{\Omega_i}(a_{n+1}) = \aleph_0$. Condition (V)(iii) is immediate by construction, as noted above. Now $(\Omega_i, T_i)_{i<j}$ is a good λ-system by Lemma 4.8 and induction hypothesis.

Finally, we define an isomorphism ψ_i from $[a, b]_{\Omega_{i+1}}$ onto $[c, d]_{\Omega_{i+1}}$ by putting $a^{\psi_i} = c$, $b^{\psi_i} = d$, $\psi_i \mid A_n \cap \Omega_i = \pi_n \mid A_n \cap \Omega_i$ and $\psi_i^{-1} \mid B_n \cap \Omega_i = \pi'_n \mid B_n \cap \Omega_i$. Then the checking of $(+)$ is straight-forward, finishing this case.

Case 2. Assume $\Lambda \neq \varnothing$ and $n = \sup \Lambda \in \Lambda$

Let $m \in \lambda$ satisfy $n \in \lambda_m$ and $n^{\phi_m} = X \in M_m$, thus $m \leqq n < i$ and $\Omega_m \subseteq \Omega_n \subseteq$

Ω_i. Here we have to define $\Omega_j \equiv \Omega_{i+1}$ and ψ_i such that, in particular, ψ_i extends the isomorphism $\psi_n : [a,b]_{\Omega_{n+1}} \to [c,d]_{\Omega_{n+1}}$. First note that whenever (A,B) is a jump of $[a,b]_{\Omega_{n+1}}$, then (A^{ψ_n}, B^{ψ_n}) is a jump of $[c,d]_{\Omega_{n+1}}$, and any jump of $[c,d]_{\Omega_{n+1}}$ is of this form. We obtain $\Omega_j \cup T_j$ by inserting points into $\Omega_{n+1} \cup T_{n+1}$ between sets A and B (A^{ψ_n} and B^{ψ_n}) for certain jumps (A,B) of $[a,b]_{\Omega_{n+1}}$. So put $\psi_i \restriction [a,b]_{\Omega_{n+1}} = \psi_n$, and let (A,B) be a jump of $[a,b]_{\Omega_{n+1}}$, thus $\sup A = \inf B \in \bar{\Omega}_{n+1} \backslash \Omega_{n+1}$. We distinguish between 4 subcases.

Subcase (*i*). We assume that no $x \in [a,b]_{\Omega_i}$ ($y \in [c,d]_{\Omega_i}$) satisfies $A < x < B$ ($A^{\psi_n} < y < B^{\psi_n}$), respectively, in Ω_i.

In this case, no point is inserted — neither between A and B nor between A^{ψ_n} and B^{ψ_n}. Observe that there exists a point $x \in T_{n+1}$ with $A < x < B$ iff there exists a point $y \in T_{n+1}$ with $A^{\psi_n} < y < B^{\psi_n}$, since Ω_{n+1} satisfies condition $(+)$.

Subcase (*ii*). We assume that there exists a point $x \in [a,b]_{\Omega_i}$ with $A < x < B$, but no $y \in [c,d]_{\Omega_i}$ with $A^{\psi_n} < y < B^{\psi_n}$.

First notice that $x \notin \Omega_{n+1}$, hence $n+1 < i$. Next we claim that there is no $z \in [c,d]_{T_i}$ with $A^{\psi_n} < z < B^{\psi_n}$. Indeed, assume there were such an element $z \in T_i$. If $z \notin T_{n+1}$, we obtain $\mathrm{Ded}(z, \Omega_{n+1}, \Omega_i) \neq \emptyset$ by $(\Omega_{n+1}, T_{n+1}) \subseteq (\Omega_i, T_i)$ (good), in contradiction to the assumption of this subcase. If, however, $z \in T_{n+1}$, let $\phi = \bar{\psi}_n$ be the extension of ψ_n to an isomorphism from $[a,b]_{\bar{\Omega}_{n+1}}$ onto $[c,d]_{\bar{\Omega}_{n+1}}$. Then $z' = z^{\phi^{-1}}$ satisfies $A < z' < B$ and $z' \in T_{n+1}$, hence $\mathrm{Ded}(z', \Omega_{n+1}, \Omega_i \cup T_i) = \{z'\}$ in contradiction to the existence of x. This shows our claim.

Now let D be a copy of the set $C = \mathrm{Ded}(x, \Omega_{n+1}, \Omega_i \cup T_i)$ and π an isomorphism from C onto D. We insert D between A^{ψ_n} and B^{ψ_n}, let the points of $(C \cap \Omega_i)^\pi$ ($(C \cap T_i)^\pi$) belong to Ω_j (T_j), respectively, and put $\psi_i \restriction C = \pi \restriction C \cap \Omega_i$.

Note that by $(\Omega_{n+1}, T_{n+1}) \subseteq (\Omega_i, T_i)$ (good) we have $\mathrm{cof}(A) = \mathrm{coi}(B) = \aleph_0$, thus $\mathrm{cof}(A^{\psi_n}) = \mathrm{coi}(B^{\psi_n}) = \aleph_0$, and $C \cap \Omega_i$, and thus also $D \cap \Omega_i$, is unbounded below and above and has countable coterminality. Furthermore, we have $\inf C$, $\sup C \in C \cap T_i$, hence $A^{\psi_n} < \min D < D \cap \Omega_j < \max D < B^{\psi_n}$ and $\min D, \max D \in T_j$. This implies (4.7)(V) for Ω_i, Ω_j, T_j, and any $a' \in D \cap \Omega_j$, since any $z \in \Omega_i$ with $z < a'$ ($a' < z$) satisfies $z < \min D$ ($\max D < z$) by the assumption of this subcase.

Subcase (*iii*). We assume that there exists a point $y \in [c,d]_{\Omega_i}$ with $A^{\psi_n} < y < B^{\psi_n}$, but no $x \in [a,b]_{\Omega_i}$ with $A < x < B$.

We deal with this case symmetrically to (ii).

Subcase (*iv*). Assume that there are points $x \in [a,b]_{\Omega_i}$, $y \in [c,d]_{\Omega_i}$ with $A < x < B$ and $A^{\psi_n} < y < B^{\psi_n}$.

Again note that $x, y \notin \Omega_{n+1}$ and $n+1<i$. We put

$$C = \mathrm{Ded}(x, \Omega_{n+1}, \Omega_i), \qquad D = \mathrm{Ded}(y, \Omega_{n+1}, \Omega_i),$$

$$a' = \inf_{\bar{\Omega}_i} C, \quad b' = \sup_{\bar{\Omega}_i} C, \quad c' = \inf_{\bar{\Omega}_i} D, \quad d' = \sup_{\bar{\Omega}_i} D.$$

Observe $a', b', c', d' \in T_i$. Obviously, $C = [a', b']_{\Omega_i}$ and $D = [c', d']_{\Omega_i}$. By $\Omega_{n+1} \subseteq \Omega_i$ (good) we obtain $\mathrm{cof}(A) = \mathrm{coi}(B) = \mathrm{cof}(A^{\psi_n}) = \mathrm{coi}(B^{\psi_n}) = \mathrm{cot}(C) = \mathrm{cot}(D) = \aleph_0$. This shows that we can deal with the intervals $[a', b']_{\Omega_i}$, $[c', d']_{\Omega_i}$ precisely as we dealt with the intervals $[a, b]_{\Omega_i}$, $[c, d]_{\Omega_i}$ in case 1, with the only exception that the endpoints of the intervals belong to T_i instead of Ω_i. Thus, both C and D are split into countably many subintervals, copies of these intervals are inserted into D and C, respectively, to obtain Ω_j, and the isomorphism ψ_i is defined accordingly.

It remains to show that if these constructions are performed for each jump (A, B) of $[a, b]_{\Omega_{n+1}}$, then our sets Ω_j, T_j satisfy the required conditions.

First note that the inserting processes have only been carried out if $n+1<i$ and in this case at most $|\mathrm{Jump}(\Omega_{n+1}, \Omega_i)| \cdot \aleph_0 < \lambda$ times, which implies $|\mathrm{Jump}(\Omega_i, \Omega_j)| < \lambda$. Now it is straight-forward to verify that Ω_j is a good λ-set, and $(\Omega_i, T_i) \subseteq (\Omega_j, T_j)$ (good) also follows immediately from our considerations. Hence $(\Omega_p, T_p)_{p<j}$ is a good λ-system by Lemma 4.8, satisfying the requirements $(+)$, $(++)$ by construction.

Case 3. Assume $\Lambda \neq \varnothing$ and $s = \sup \Lambda \notin \Lambda$

Obviously, s is a limit-ordinal and $s \leqq i$. Observe $\Omega_s = \bigcup_{n<s} \Omega_n$, $T_s = \bigcup_{n<s} T_n$, and whenever $n, m \in \Lambda$ satisfy $n<m$, then $\psi_m \mid [a, b]_{\Omega_{n+1}} = \psi_n$ by induction hypothesis and $(++)$. This enables us to define an isomorphism $\psi : [a, b]_{\Omega_s} \to [c, d]_{\Omega_s}$ satisfying $\psi \mid [a, b]_{\Omega_{n+1}} = \psi_n$ for each $n \in \Lambda$, such that, if we extend ψ to an isomorphism $\bar{\psi} : [a, b]_{\bar{\Omega}_s} \to [c, d]_{\bar{\Omega}_s}$, then

$$(T_s \cap [a, b]_{\bar{\Omega}_s})^{\bar{\psi}} = T_s \cap [c, d]_{\bar{\Omega}_s}.$$

Now we may continue precisely as in Case 2, only dealing with Ω_s, T_s, ψ instead of Ω_{n+1}, T_{n+1}, ψ_n, in order to obtain a good λ-set Ω_j, a set $T_j \subseteq \bar{\Omega}_j \setminus \Omega_j$ and an isomorphism $\psi_i : [a, b]_{\Omega_j} \to [c, d]_{\Omega_j}$, such that $(\Omega_s, T_s) \subseteq (\Omega_j, T_j)$ (good), $\psi_i \mid [a, b]_{\Omega_s} = \psi$ and

$$(T_j \cap [a, b]_{\bar{\Omega}_j})^{\bar{\psi}_i} = T_j \cap [c, d]_{\bar{\Omega}_j}.$$

Hence again $(\Omega_p, T_p)_{p \leqq j}$ is a good λ-system, completing this case and Step 3.

Part III. The chain $(\Omega, \leqq)$ *is doubly homogeneous and satisfies* $\cot(a) = \aleph_0$ *for each* $a \in \Omega$

Now suppose our good λ-system $(\Omega_i, T_i)_{i\in\lambda}$ is constructed as prescribed in Parts I and II. We put $\Omega = \bigcup_{i\in\lambda} \Omega_i$, $T = \bigcup_{i\in\lambda} T_i$ and $(\Omega \cup T, \leqq) = \bigcup_{i\in\lambda}(\Omega_i \cup T_i, \leqq)$. First we want to show that $(\Omega, \leqq)$ is a doubly homogeneous chain.

Indeed, let $a, b, c, d \in \Omega$ satisfy $a < b$ and $c < d$. By Remark 2.6, we have to construct an isomorphism $\psi : [a, b]_\Omega \to [c, d]_\Omega$. Since Ω is unbounded, we may assume $b < c$ w.l.o.g. Choose $m \in \lambda$ such that $a, b, c, d \in \Omega_m$, and let I be the set of all $i \in \lambda$ such that $i \in \lambda_k$ for some $k \in \lambda$ with $k \geqq m$ and $i^{\phi_k} = (a, b, c, d)$. Note that $|I \cap \lambda_k| = 1$ for each $k \in \lambda$ with $k \geqq m$, hence $|I| = \lambda$ and I is cofinal in λ. Thus $\Omega = \bigcup_{i\in I}\Omega_{i+1}$, and for each $i \in I$ there exists an isomorphism $\psi_i : [a, b]_{\Omega_{i+1}} \to [c, d]_{\Omega_{i+1}}$ such that whenever $i, j \in I$ satisfy $j < i$, then $\psi_i \,|\, [a, b]_{\Omega_{j+1}} = \psi_j$. Hence we may define the required isomorphism ψ simply by putting $\psi \,|\, [a, b]_{\Omega_{i+1}} = \psi_i$ for each $i \in I$.

Secondly we show that $\cot(a) = \aleph_0$ for each $a \in \Omega$. Indeed, let $i \in \lambda$ and $a \in \Omega_i$. Then $\cot_{\Omega_i}(a) = \aleph_0$ since $(\Omega_i, \leqq)$ is a good λ-set. Now Lemma 4.6 implies $\cot_\Omega(a) = \aleph_0$.

Part IV. We claim that we may perform our Construction 2.16 such that $(T_L, \leqq_L) = (T_l(\Omega), \leqq_l)$, $(T_R, \leqq_R) = (T_r(\Omega), \leqq_r)$

W.l.o.g. we only show (by transfinite induction) that $(T_R, \leqq_R)$ may be taken as the tree $(T_r(\Omega), \leqq_r)$, i.e. satisfies the construction requirements of (2.16). Put $m := \max(T_{\tilde{R}}, \leqq) = \min(T_R, \leqq_R)$. Observe that indeed $T_{\tilde{R}} \setminus \{m\} \subseteq T_0 \subseteq T \subseteq \hat{\Omega}$ by Lemma 4.6(b) or as in the proof of (4.10). In the following, we will have to distinguish carefully between the orders $(T_R, \leqq_R)$, $(T_r(\Omega), \leqq_r)$ which are trees and the chains $(T_{\tilde{R}} \setminus \{m\}, \leqq) \subseteq T_0 \subseteq (\Omega \cup T, \leqq)$. As in Remark 4.11, we put $T_{\tilde{R}i} = \{x \in T_{\tilde{R}} \,|\, (\{y \in T_R \,|\, y <_R x\}, \leqq_R) \cong i\}$ for each ordinal i; let h be the least ordinal α such that $T_{\tilde{R}\alpha} = \varnothing$, thus $T_{\tilde{R}} = \bigcup_{i<h} T_{\tilde{R}i}$. For $A \subseteq T_R$, M_A again denotes the set of all minimal elements of $(T_R, \leqq_R)$ above A.

Step 1. Start of the induction

First notice that we have $m > T_0$, hence $m > \Omega_0 \cup T_0$ according to our construction of (Ω_0, T_0) (see Part II, Step 1). By $\Omega_0 \subseteq \Omega_j$ (good) for $1 \leqq j \in \lambda$ and Lemma 4.6, Ω_0 is unbounded above in Ω. Hence we may identify $m = \infty \in \hat{\Omega}$. Next we choose $\infty' \in \bar{\Omega}$ such that $\infty' < \min(T_{\tilde{R}}, \leqq)$. This is in harmony with $T_R \in \mathcal{T}_R$, i.e. $m^\Phi = 3$, cf. Definition 2.17. Furthermore, if $T_R = \{\infty\}$, we have $\operatorname{cof}(\Omega_0) = \aleph_0$ by the construction in Part II, Step 1, hence $\operatorname{cof}(\Omega) = \aleph_0$ by $m = \sup \Omega_0 = \sup \Omega$ in $\hat{\Omega}$, and thus $T_R = T_r(\Omega)$ is established in this special case.

Step 2. Induction step from i to $i+1$

Next let $i<h$ and $a \in T_{\tilde{R}i}$, and assume $a'<a$, $a^{\Phi}=3$ (cf. (2.16); the case $a<a'$, $a^{\Phi}=1$ is dealt with symmetrically). We distinguish between the cases $M_a=\emptyset$ and $M_a \neq \emptyset$.

First assume $M_a=\emptyset$; we claim $\mathrm{cof}(a)=\aleph_0$ in $\hat{\Omega}$ (thus by (2.16) a has to be a maximal element in $T_r(\Omega)$, in accordance with $M_a=\emptyset$). If $a=\infty$, this was shown above. Hence assume $a\neq\infty$ now. By Remark 4.12(d) there exists a maximal element b in $(T_{\tilde{R}}, \leqq)$ with $b<a$. Now by Part II, Step 1 the set $\{x\in\Omega_0 \mid b<x<a\}$ is a good λ-set of countable coterminality. This shows $\mathrm{cof}_{\hat{\Omega}}(a)=\mathrm{cof}_{\hat{\Omega}_0}(a)=\aleph_0$ by Lemma 4.6, as claimed.

Next assume $M_a\neq\emptyset$. By Remark 4.12(c), we have $M_a<a$ in $(T_{\tilde{R}}, \leqq)$, furthermore $(M_a, \leqq)$ is an uncountable well-ordered set isomorphic to $|M_a|$, and $M_a\cup\{a\}$ is closed in $T_0\cup\{\infty\}\subseteq\hat{\Omega}_0$ such that $a=\sup M_a$ in $\hat{\Omega}_0$. By Lemma 4.6, $M_a\cup\{a\}$ is also closed in $\hat{\Omega}$ such that $a=\sup M_a$ in $\hat{\Omega}$. This shows $(M_a, \leqq)\cong\mathrm{cof}(a)\neq\aleph_0$. We may assume $a'<M_a$ by induction hypothesis; now choose, for each $x\in M_a$, an element $x'\in\bar{\Omega}$ with $x<x'<\{z\in M_a \mid x<z\}$ and also $\max(M_x, \leqq)<x'$ if $M_x\neq\emptyset$. Note that $a^{\Phi}=3$ implies $x^{\Phi}=1$ for each $x\in M_a$ which is in accordance with $x<x'$.

Hence in both of these cases the set $(M_a, \leqq)$ satisfies the construction requirements in (2.16).

Step 3. Induction step for limit-ordinals

Here let $i<h$ be a limit-ordinal, P a maximal path in $\bigcup_{j<i} T_{\tilde{R}i}$, and $P_n=\{a\in P \mid a^{\Phi}=n\}$ $(n=1,3)$. If $a\in P_1$ $(a\in P_3)$, let a^+ be the minimal element in P above a, i.e. $\{a^+\}=P\cap M_a$, hence $a^+\in P_3$ $(a^+\in P_1)$ by $a^{\Phi}=1$ $(a^{\Phi}=3)$, and $a<a^+$ $(a^+<a)$; we put $V_a=[a,a^+]$ $(V_a=[a^+,a])$, respectively. Note that $V_a\supseteq V_b$ whenever $a,b\in P$ and $a<_R b$ (cf. (4.12)(b)). We have either $M_P=\{a_P\}$ with $a_P^{\Phi}=2$ or $M_P=\{a_P,b_P,c_P\}$ with $a_P^{\Phi}=1$, $b_P^{\Phi}=2$, $c_P^{\Phi}=3$. If $M_P=\{a_P\}$, we have $P_1<a_P<P_3$ and $a_P=\sup P_1=\inf P_3$ in T_0 by Remark 4.12(e), hence also in $\bar{\Omega}$ by Lemma 4.6, and thus $\bigcap_{a\in P} V_a=\{a_P\}$. We put $a'_P=a_P$ according to (2.16) which is in harmony with $a_P^{\Phi}=2$.

If, on the other hand, $|M_P|=3$, we have $P_1<a_P<c_P<P_3$ and $a_P=\sup P_1$, $c_P=\inf P_3$ in T_0 by Remark 4.12(e), hence again also in $\bar{\Omega}$ by (4.6), and thus $\bigcap_{a\in P} V_a=[a_P,c_P]$. By $a_P^{\Phi}=1$ $(c_P^{\Phi}=3)$, $(M_{a_P}, \leqq)$ $((M_{c_P}, \leqq))$ is inversely well-ordered (well-ordered), if it is non-empty, in which case we put $\bar{a}=\max(M_{a_P}, \leqq)$ $(\bar{c}=\min(M_{c_P}, \leqq))$, respectively. If M_{a_P} (M_{c_P}) is empty, put $\bar{a}=a_P$ $(\bar{c}=c_P)$. In any case we have $a_P\leqq\bar{a}<\bar{c}\leqq c_P$. Now choose $a'_P, c'_P\in\bar{\Omega}$ and an element $x_P\in\bar{\Omega}$ with $a_P\leqq\bar{a}<a'_P<x_P<c'_P<\bar{c}\leqq c_P$ in harmony with $a_P^{\Phi}=1$,

$c_P^\Phi = 3$. We have $b_P \in T_R \backslash T_{\tilde{R}}$ and $b_P \not\in \bar{\Omega}$, but here we may identify b_P with x_P and put $b'_P = b_P$ in harmony with $b_P^\Phi = 2$. Hence the construction requirements of (2.16) for limit steps are also satisfied. This finishes the proof that $(T_R, \leqq_R) = (T_r(\Omega), \leqq_r)$.

With this, Theorem 2.11 is proved.

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